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Contribution of Exponential and Integral Functions in some Probability Distributions

Abd Elmoniem Awad Elkariem

Faculty of Engineering, Nile Valley University
Corresponding Author: admin@expert.net

Abstract

This article reviews and interprets exponential and integral functions on representations of some probability distributions. The literature survey covers Poisson distribution, Normal Distribution, Exponential Distribution, Gamma Distribution, and Weibull Distribution. The article explained in detail the contribution of both functions on the chosen probability distributions. The most important recommendation for those who want to be good statisticians is that they must have a good background in mathematics.
Keywords: Exponential, Gamma, Beta, Contribution

مساهمة الدالة الأسية والدالة التكاملية في بعض التوزيعات الاحتمالية

عبد المنعم عوض الكريم

كلية الهندسة، جامعة وادي النيل

المؤلف المرسل: admin@expert.net

مُسْتَخْلَص

تناولت هذه الورقة دور الدالتين الأسية والتكاملية في التوزيعات الاحتمالية، غطي المسح النظري توزيع بواسون، التوزيع الطبيعي توزيع قاما وتوزيع ويبيل وتوضح بشيء من التفصيل مساهمة كل من الدالتين في انتاج تلك التوزيعات الاحتمالية المختارة، أهم التوصيات تحفيز الذين يرغبون في أن يكونوا إحصائيات بكفاءة عالية عليهم أن يحصلوا على خلفية مناسبة في الرياضيات.

كلمات مفتاحية: الأسية، قاما، بيتا، مساهمة

The overall Aim

The overall aim of this article is to display the contribution of exponential and integral functions in some probability distributions, and to stimulate those who want to be a good statistician i.e. they should be familiar with these functions as well as other functions.

Introduction

Statistics provides models that are needed to study situations involving uncertainties, in the same way as calculus provide models that are needed to describe, say Newton Laws of motions. The names which are connected most prominently with the growth of mathematical statistics are R.A. Fisher, J. Neyman, E.S. Peorson and A. Wald (John and Ronold, 1980, P:2). R.A. Fisher who got a degree of Doctor of Science at University of Evitcago in 1952 (George, 2014, P:1) , his name is connected with F distribution , George E.P.Box concluded in his article some questions about Fisher which might be asked :

- Was he an applied statistician?
- Was he a mathematical statistician?
- Was he data analyst?
- Was he a designer of investigation?

It is surely because he was all of these he was much more than the sum of the part . He provides an example we can seek to follow (George, 2014, P:2).Some of mathematical topics have useful background to statistics and probability distributions : They are Boolean algebra , calculus , functions , mathematical transforms , and matrices . (E.Frend & Waple, 1980, P:177) claimed that “ The beginnings of the mathematics of statistics may be found in mid eighteenth – century studies in probability motivated by interest in games of chance. The theory thus developed for ″head or tails ″ or ″ red or black ″soon found applications in situations where the outcomes were ″boy or girl ″ or ″life or death″ or ″pass or fail″ and scholars began to apply probability theory to actuarial problems and some aspects of the social sciences and physics.

Literature of Some Probability Distributions

Exponential Functions

The origin of Exponential Function is the Binomial theorem is stated as:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Replacing x by $\frac{1}{n}$

$$\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}} = 1 + n\left(\frac{1}{\mathrm{n}}\right) + \frac{n(n-1)\left(\frac{1}{\mathrm{n}}\right)^2}{2!} + \frac{n(n-1)(n-2)\left(\frac{1}{\mathrm{n}}\right)^3}{3!} + \cdots$$

$$\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}} = 1 + 1 + \frac{\left(\frac{\mathrm{n}}{\mathrm{n}}\right)(1-\left(\frac{1}{\mathrm{n}}\right))}{2!} + \frac{\left(\frac{\mathrm{n}}{\mathrm{n}}\right)(1-\left(\frac{1}{\mathrm{n}}\right))(1-\left(\frac{2}{\mathrm{n}}\right))}{3!} + \cdots$$

Taking the limit as $\mathrm{n} \to \infty$

$$\lim_{\mathrm{n}\to\infty}\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}} = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

= e (known as a magic number)

If we rase $\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$ to x , then $\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{nx}}$ and taking the limit as $\mathrm{n} \to \infty$

$$\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{nx}} = 1 + nx\left(\frac{1}{\mathrm{n}}\right) + \frac{nx(nx-1)\left(\frac{1}{\mathrm{n}}\right)^2}{2!} + \frac{nx(nx-1)(nx-2)\left(\frac{1}{\mathrm{n}}\right)^3}{3!} + \cdots$$

$$\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}x} = 1 + \frac{\left(\frac{\mathrm{n}}{\mathrm{n}}\right)x}{1!} + \frac{\left(\frac{\mathrm{n}x}{\mathrm{n}}\right)(x-\left(\frac{1}{\mathrm{n}}\right))}{2!} + \frac{\left(\frac{\mathrm{nx}}{\mathrm{n}}\right)(x-\left(\frac{1}{\mathrm{n}}\right))(x-\left(\frac{2}{\mathrm{n}}\right))}{3!} + \cdots$$

Taking the limit as $\mathrm{n} \to \infty$

$$\lim_{\mathrm{n}\to\infty}\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{nx}} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Which known as Exponential function (KA Stroud, 2004, P:4). The Exponential function plays important roles in many probability distributions such as: Poisson Distribution, Normal Distribution, Exponential Distribution and Weibull Distribution.

Poisson Distribution

Poisson distribution can be represented from the known Binomial distribution specifically, we shall investigate the limiting from the Binomial distribution whenn $\rightarrow \infty$ and p $\rightarrow 0$, while np remains constant be λ , that is $np = \lambda$, and hence $p = \frac{\lambda}{n}$

That could be explained as follows:

$$b(x, n, p) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^{x}\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^{x}\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Then one of x factors should be divided in $\left(\frac{\lambda}{n}\right)^{x}$ into each factor of the product

$n(n-1)(n-2)\ldots\ldots(n-x+1)$ and write

$\left(1 - \frac{\lambda}{n}\right)^{n-x}$as:$\left(\left(1 - \frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right)^{-\lambda}\left(1 - \frac{\lambda}{n}\right)^{-x}$

$$\frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{x-1}{n}\right)}{x!}(\lambda)^{x}\left(\left(1 - \frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right)^{-\lambda}\left(1 - \frac{\lambda}{n}\right)^{-x}$$

Finally, if we let n $\rightarrow \infty$ while $x$and λ remain fixed, we find that the limit becomes:

$$\frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{x-1}{n}\right)}{x!} \rightarrow 1$$

$$\left(1 - \frac{\lambda}{n}\right)^{-x} \rightarrow 1$$

$$\left(1 - \frac{\lambda}{n}\right)^{\frac{-n}{\lambda}} \rightarrow e$$

and hence, that the Limiting distribution becomes:

$$p(x, \lambda) = \frac{\lambda^{x}e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, \ldots.$$

which is a Poisson distribution this distribution, had been named after the French mathematician Simeon Poisson (1781 – 1840) (Susan and Jesse, 2003, P:9)

The Integral Functions

Some function are most conveniently defined in the form of integrals such as Gamma function and Beta function (Stroud, 2004, P:125)

The Gamma Function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \text{ and is convergent for } x > 0$$

If we replace x by $x+1$

$$\text{Then } \Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\, dt$$

Integrating by parts:

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty + x\int_0^\infty e^{-t}\ t^{x-1}\, dt$$

$$= [0-0] + x\,\Gamma(x)$$

$$\therefore\ \Gamma(x+1) = x\,\Gamma(x)$$

Put $n = x$

$$\Gamma(n+1) = n\,\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$= (n-1)(n-2)\,\Gamma(n-2)$$

$$= (n-1)(n-2)\ \ldots\ 1\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = -\left[e^{-t}\ \right]_0^\infty = 0 + 1 = 1$$

$$\Gamma(n+1) = n(n-1)(n-2) \times 1 = n!$$

$$\therefore\ \Gamma(n+1) = n!$$

For Example: $\Gamma(7) = 6! = 720$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

The Beta Function

The Beta function $B(m, n)$ is defined by :

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

which converges for $m > 0$ and $n > 0$

putting $(1 - x) = \mu \quad \therefore x = 1 - \mu$

$$dx = -d\mu$$

Limits when $x = 0, \mu = 1, when\ x = 1, \mu = 0$

$$B(m, n) = -\int_1^0 (1-\mu)^{m-1}\mu^{n-1}\, d\mu = \int_0^1 (1-\mu)^{m-1}\mu^{n-1}\, d\mu$$

$$= B(n, m)$$

$$\therefore B(m, n) = B(n, m)$$

The Relation between gamma and beta functions:

$$B(m, n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}$$

The Normal distribution

A random variable has a normal distribution and it is referred to as a normal random variable if and only if its probability density is given by: (John and Ronold, 1980, P:5)

$$N(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{for } -\infty < x < \infty \text{ where } \sigma > 0$$

Where σ is the standard deviation and μ is the mean of the distribution, we need to show that the total area from $-\infty\ to\ \infty\ is\ 1$, making the substitution $z = \frac{x-\mu}{\sigma}$ we get:

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$\text{Since } \int_0^{\infty} e^{-\frac{1}{2}z^2}\, dz = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\text{Then } \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{\pi}}{\sqrt{2}} = 1$$

Moment Generating Function

The exponential function plays an impartment role is evaluating the moment generating function for the probability distribution, the two expectations$E[x] and\ E[x^2]$ are very useful in determine the mean and the variance of the distributions.

Definition

Let x be a random variable with Discrete probability distribution the moment generating function for x is denoted by: $m_x(t) = E[e^{tx}]$

Gamma Distribution

The theoretical basis for gamma distribution is the gamma function (KA Stroud,2004,P:136)

Definition

A random variable x with density:

$$f(x) = \frac{1}{\Gamma(\propto)\beta^n} x^{\propto-1} e^{-x/\beta}$$

$$x > 0\,, \propto > 0\,, \beta > 0$$

is said to have a gamma distribution with parameter $\propto$ and β with moment generating as :

$$m_x(t) = (1-\beta t)^{-\propto} t > \frac{1}{\beta}$$

and mean = $\propto \beta$, variance = $\propto \beta^2$

proof:

by definition:

$$m_x(t) = E[e^{tx}]$$

$$= \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\propto)\beta^{\propto}} x^{\propto-1} e^{-x/\beta}\, dx \qquad (1)$$

$$= \frac{1}{\Gamma(\propto)\beta^{\propto}} \int_{0}^{\infty} x^{\propto-1} e^{-(\frac{1}{\beta}-t)x} \quad dx \qquad (2)$$

$$Let\ z = (\ 1 - \ \beta t)\frac{x}{\beta}$$

$$x = \beta z(\ 1 - \ \beta t)$$

$$dx = \beta dz(\ 1 - \ \beta t) \qquad (3)$$

Substituting in (2)

$$\mathrm{m_x(t)} = \frac{1}{\Gamma(\propto)\beta^{\propto}} \int_{0}^{\infty} (\frac{\beta z}{1 - \ \beta t})^{\propto-1} \frac{e^{-z}\beta dz}{(1 - \ \beta t)}$$

$$= \frac{1}{\Gamma(\propto)\beta^{\propto}} \frac{\beta^{\propto}}{(1 - \ \beta t)^{\propto}} \int_{0}^{\infty} z^{\propto-1}\, e^{-z}\, dz \qquad (4)$$

$$Sice\colon\ \Gamma(\propto) = \int_{0}^{\infty} z^{\propto-1}\, e^{-z}\, dz$$

$$\text{Then:}\ \mathrm{m_x(t)} = \frac{1}{\Gamma(\propto)\ (1 - \ \beta t)^{\propto}} \times \ \Gamma(\propto)$$

$$= (\frac{1}{1 - \ \beta t})^{\propto} \quad = \ (1 - \ \beta t)^{-\propto}$$

$t < \frac{1}{\beta}$ to avoid division by zero

To find then mean:

$$u = \mathrm{E}[x] = \frac{d}{dt}(\mathrm{m_x(t)})$$

$$= \frac{d}{dt}\ (1 - \ \beta t)^{-\propto}{}_{\mathrm{t=0}}$$

$$= - \propto \ (1 - \ \beta t)^{-(\propto+1)} \times -\beta$$

$$= \propto \beta$$

$$var(x) = \mathrm{E}[x^2] - E[x]^2$$

$$E[x^2] = \frac{d^2}{dt^2}\left(m_x(t)\right) = \frac{d^2}{dt^2}\ (1-\ \beta t)^{-\propto}{}_{t=0}$$

$$= \propto (\propto +1)\ (1-\ \beta t)^{-\propto -2} \times -\beta_{t=0}$$

$$= \propto (\propto +1)\ \beta^2$$

$$var(x) = \propto (\propto +1)\ \beta^2 -\ (\propto \beta)^2$$

$$=\ \propto^2 \beta^2 +\ \propto \beta^2 -\ \propto^2 \beta^2 =\ \ \propto \beta^2$$

Hence the mean and variance of gamma distribution as:

$$\mu = \propto \beta \ and\ var(x) =\ \propto \beta^2$$

The Beta Distribution

Definition

A random variable X has a beta distribution and it is referred to as beta random variable if and only if:

$$f(x) = \begin{cases} \dfrac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} x^{\propto -1}(1-x)^{\beta-1} & \text{for} \quad 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

We need to show that:

$$\int_0^1 \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} x^{\propto -1}(1-x)^{\beta-1}\ \ dx \ \ = \ \ 1$$

That is:

$$\int_0^1 x^{\propto -1}(1-x)^{\beta-1}\ \ dx\ = \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\propto +\beta)}$$

$$= \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \int_0^1 x^{\propto -1}(1-x)^{\beta-1}\ \ dx$$

$$= \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \times \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\propto +\beta)} \ \ = \ \ 1$$

The mean and the variance of beta distribution*:*

$$the\ mean:\ \ \mu\ =\frac{\alpha}{\propto+\beta}$$

$$\sigma^2\ =\ \frac{\alpha\beta}{(\propto+\beta)^2(\propto+\beta+1\ )}$$

By definition: The mean is

$$\mu\ =\ \mathrm{E}[x]$$

$$=\frac{\Gamma(\propto+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\int_0^1 x^{\propto-1}(1-x)^{\beta-1}\ \ \mathrm{dx}$$

$$=\frac{\Gamma(\propto+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\int_0^1 x^{\propto-1}(1-x)^{\beta-1}\ \ \mathrm{dx}$$

Since

$$\int_0^1 x^{\propto-1}(1-x)^{\beta-1}\ \ \mathrm{dx}\ \ =\ B(\propto,\beta\ )$$

and

$$B(\propto+1\,,\beta\ )=\ \frac{\Gamma(\alpha+1)\ \Gamma(\beta)}{\Gamma(\propto+\beta+1)}$$

$$=\ \frac{\alpha\ \Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\propto+\beta+1)}$$

$$\mu\ =\ \frac{\Gamma(\propto+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\int_0^1 x^{\propto-1}(1-x)^{\beta-1}\ \ \mathrm{dx}\ \ =\ \frac{\Gamma(\alpha+1)\ \Gamma(\beta)}{\Gamma(\propto+\beta+1)}\times\frac{\Gamma(\propto+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}$$

$$\therefore\ \ \mu\ =\ \frac{\Gamma(\propto+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\times\ \frac{\alpha\ \Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\propto+\beta)(\propto+\beta)}=\ \frac{\alpha}{\propto+\beta}$$

$$\sigma^2=\ \mathrm{E}[x^2]\ \ -\ \ \mathrm{E}[x]^2$$

$$\mathrm{E}[x^2]=\int_0^1\frac{\Gamma(\propto+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}x^{\propto+1}(1-x)^{\beta-1}\ \ \mathrm{dx}$$

$$Since:\ \int_0^1 x^{\propto+1}(1-x)^{\beta-1}\ \ \mathrm{dx}\ \ =\ B(\propto+2\,,\beta\ )$$

$$and\ B(\propto +2\,,\beta\,) = \frac{\Gamma(\alpha+2)\,\Gamma(\beta)}{\Gamma(\propto +2+\beta)}$$

$$= \frac{(\alpha+1)\,\Gamma(\alpha+1)\,\Gamma(\beta)}{(\propto +\beta+1)\,\Gamma(\propto +\beta+1)}$$

$$= \frac{\alpha(\alpha+1)\,\Gamma(\alpha)\,\Gamma(\beta)}{(\propto +\beta+1)(\propto +\beta)\,\Gamma(\propto +\beta)}$$

$$\mathrm{E}[x^2] = \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \times \frac{\alpha(\alpha+1)\,\Gamma(\alpha)\,\Gamma(\beta)}{(\propto +\beta+1)(\propto +\beta)\,\Gamma(\propto +\beta)} = \frac{\alpha(\alpha+1)}{(\propto +\beta+1)(\propto +\beta)}$$

$$\sigma^2 = E[x^2] - E[x]^2 = \frac{\alpha(\alpha+1)}{(\propto +\beta+1)(\propto +\beta)} - \frac{\propto^2}{(\propto +\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\propto +\beta) - \propto^2 (\propto +\beta+1)}{(\propto +\beta+1)(\propto +\beta)^2}$$

$$= \frac{\propto^3 + \propto^2 \beta + \propto^2 + \propto \beta - \propto^3 - \propto^2 \beta - \propto^2}{(\propto +\beta+1)(\propto +\beta)^2}$$

$$\therefore \sigma^2 = \frac{\alpha\beta}{(\propto +\beta)^2(\propto +\beta+1\,)}$$

The mean and variance of beta distribution are:

$$\mu = \frac{\alpha}{\propto +\beta} \quad and\ \sigma^2 = \frac{\alpha\beta}{(\propto +\beta)^2(\propto +\beta+1\,)}$$

The Weibull Distribution

Definition

A random variable x is said to have a Weibull distribution with parameter α and β if its density is given by: (Stroud, 2004, P:129)

$$f(x) = \propto \beta\ x^{\beta-1} e^{-\propto x^{\beta}}\,, x > 0\,, \propto > 0\,, \qquad \beta > 0$$

The mean of distribution is:

$$\mu = \alpha^{-\frac{1}{\beta}}\ \Gamma(1+\frac{1}{\beta})$$

Proof :

$$\mu = \mathrm{E}[x]$$

$$= \int_0^{\infty} x\ \propto \beta\ x^{\beta-1} e^{-\propto x^{\beta}}\ \mathrm{dx}$$

$$= \int_0^\infty \propto \beta \;\; x^{\beta} e^{-\propto x^{\beta}} \;\; \mathrm{dx}$$

$$Let: \quad z = \propto x^{\beta}$$

$$x = (\frac{z}{\alpha})^{\frac{1}{\beta}}$$

$$dx = \frac{1}{\beta}(\frac{z}{\alpha})^{\frac{1}{\beta}-1} dz$$

$$Then: \quad \mathrm{E}[x] = \int_0^\infty \propto \beta \, (\frac{z}{\alpha}) \, e^{-z} (\frac{1}{\alpha\beta})(\frac{z}{\alpha})^{\frac{1}{\beta}-1} \;\; \mathrm{dz}$$

$$= \int_0^\infty (\frac{z}{\alpha})^{\frac{1}{\beta}} e^{-z} \;\; \mathrm{dz}$$

$$= \alpha^{-\frac{1}{\beta}} \int_0^\infty z^{\frac{1}{\beta}} e^{-z} \;\; \mathrm{dz}$$

$$Since: \quad \int_0^\infty z^{\frac{1}{\beta}} e^{-z} \;\; \mathrm{dz} \quad = \quad \Gamma(1 + \frac{1}{\beta})$$

$$\therefore \; \mathrm{E}[x] = \mu = \alpha^{-\frac{1}{\beta}} \, \Gamma(1 + \frac{1}{\beta})$$

$$\sigma^2 = \mathrm{E}[x^2] - \mathrm{E}[x]^2$$

$$\mathrm{E}[x^2] = \int_0^\infty x^{\quad 2} \propto \beta \;\; x^{\beta-1} e^{-\propto x^{\beta}} \;\; \mathrm{dx}$$

$$= \int_0^\infty \propto \beta \;\; x^{\beta+1} e^{-\propto x^{\beta}} \;\; \mathrm{dx}$$

$$Let: \; z = \propto x^{\beta}$$

$$x = (\frac{z}{\alpha})^{\frac{1}{\beta}}$$

$$dx = (\frac{1}{\alpha\beta})(\frac{z}{\alpha})^{\frac{1}{\beta}-1} dz$$

$$\mathrm{E}[x^2] = \int_0^{\infty} \propto \beta \, (\frac{z}{\alpha})^{1+\frac{1}{\beta}} \frac{e^{-z}}{\alpha\beta} (\frac{z}{\alpha})^{\frac{1}{\beta}-1} \quad \mathrm{dz}$$

$$= \int_0^{\infty} (\frac{z}{\alpha})^{\frac{2}{\beta}} e^{-z} \;\; \mathrm{dz}$$

$$= (\frac{1}{\alpha})^{\frac{2}{\beta}} \int_0^{\infty} z^{\frac{2}{\beta}} e^{-z} \;\; \mathrm{dz}$$

$$= \alpha^{-\frac{2}{\beta}} \Gamma\left(\frac{2}{\beta} + 1\right)$$

$$\sigma^2 = \mathrm{E}[x^2] - [x]^2$$

$$= \alpha^{-\frac{2}{\beta}} \Gamma(\frac{2}{\beta} + 1) - \left(\alpha^{-\frac{1}{\beta}} (\Gamma 1 + \frac{1}{\beta})\right)^2$$

$$\therefore \sigma^2 = (\frac{1}{\alpha})^{\frac{2}{\beta}} \Gamma(\frac{2}{\beta} + 1) - (\frac{1}{\alpha})^{\frac{2}{\beta}} \left(\Gamma\left(1 + \frac{1}{\beta}\right)\right)^2$$

Discussion

The origin of Exponential function, Gamma and beta functions has been introduced. Grasping those functions helps to understand the behaviour of the probability distributions. Looking at Poisson distributions it could be seen that it is impossible to determine the probability function without the exponential function. The Poisson distribution function is written as: (Sanders and Smidt, 2000, pp 177-178)

$$p\,(x\,,\lambda) = \frac{\lambda^x \mathrm{e}^{-\lambda}}{x!} \; for \; x = 0\,,1\,,2\,,3, \ldots$$

Some authors tended to evaluate $\mathrm{e}^{-\lambda}$ to make life easy: They constructed table for e^{x} and e^{-x} one of the mis Frenund in his book (John and Ronold, 1980, P:16)

The gamma function reduces a lot of work specially in Integrating some difficult integral of the from:

$$\int_0^\infty t^n e^{-t} \text{ dt}$$

and also, the evaluation of $\Gamma(\frac{1}{2})$ which has been use in proving the function of standard normal function which severs as a probability density function.

The Beta function involved in the Beta distribution function,helps to find the mean and the variance of the distribution.

Conclusion

In this article, the exponential and integral functions have been displayed.

The roles in some probability distributions have been explained. Those who intend to be good statisticians are recommended to have good background in mathematics.

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