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A sharp Trudinger-Moser Type Inequality for Unbounded Domains and Higher Order Derivatives

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Abstract

The Trudinger- Moser inequality states that for functions $u \in H_o^{1,n}(\Omega)$, of bounded domain Ω with $\int |\nabla u| \, dx \leq 1$ one has $\lim_{k\to+\infty} \int_{B_1} (e^{\beta u_k^{\frac{n}{n-1}}} - 1)\, dx \leq c\, |\Omega|$, with c independent of u. It is shown that for $n = 2$ the bound $c|\Omega|$ may be replaced by a uniform constant d independent of Ω if the Drichlet norm is replaced by the Sobolev norm.In this paper the results for $n > 2$ have been showed with a lower bound and gradient estimate.

Keywords: *Truding-Moser inequality, blow-up analysis, best constant, unbounded domain, Mathematics subject classification.*

متباينة ترودنقر. موزار الحادة للمجالات غير المحدودة وللمشتقات عالية المستوى

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قسم الرياضيات، كلية المعلمين، جامعة وادي النيل، عطبرة، السودان

المؤلف:

مُسْتَخْلَص

وُجد ان متباينة ترودنقر. موزار تقرر انه لكل الدوال $u \in H_o^{1,n}(\Omega)$, والتي مجالها Ω (وهو مجال محدود) مع التكامل $\int |\nabla u| dx \leq 1$ لها النهاية $\lim_{k \to +\infty} \int_{B_1} (e^{\beta u_k^{\frac{n}{n-1}}} - 1)\, dx \leq c\, |\Omega|$ عندما يكون c مستقل عن u. وهذا واضح عند n=2 أن الحد $c|\Omega|$ يمكن استبداله بالثابت المنتظم d مستقلا عن Ω .عند استبدال نظيم در شلت بنظيم سو بوليف.عُرضت في هذه الورقة نتائج من أجل n >2 مع الحد الأقل وتقديرات الانحدار.

كلمات مفتاحية:*متباينة ترودنقر. موزار، تحليل الانشطار، أفضل ثابت، المجال غير المحدود، تصنيف المواضيع الرياضية.*

Introduction

Let $H_o^{1,p}(\Omega), \Omega \subseteq R^n$, be the usual Sobolev space. *i.e.* the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{H^{1,p}(\Omega)} = \left(\int_\Omega \left(|\nabla u|^p + |u|^p\right)dx\right)^{\frac{1}{p}}.$$

It is well-known that

$$H_o^{1,p}(\Omega) \subset L^{\frac{pn}{n-p}}(\Omega) \quad if \quad 1 \le p < n$$

$$H_o^{1,p}(\Omega) \subset L^\infty(\Omega) \quad if \quad n < p$$

The case $p = n$ is the limit case of these embeddings and it is known that

$$H_o^{1,p}(\Omega) \subset L^q(\Omega) \quad for \quad n \le q < +\infty$$

When Ω is a bounded domain, we usually use the Drichlet norm $\|u\|_D =$ $\left(|\nabla u|^n dx\right)^{\frac{1}{n}}$ in place of $\|\cdot\|_{H^{1,n}}$. In this case we have the famous Trudinger-Moser inequality (see [11], [4], [9]) for the limit case $p = n$ which states that

$$\sup_{\|u\|_D \le 1} \int_\Omega \left(e^{\propto |u|^{\frac{n}{n-1}}} - 1\right)dx = c(\Omega, \propto)\begin{cases} < +\infty & when \quad \propto \le \propto_n \\ = +\infty & when \quad \propto > \propto_n \end{cases} \quad (1)$$

where $\propto = n\,\omega_{n-1}^{\frac{1}{n-1}}$, and ω_{n-1} is the measure of unit sphere in R^n.The Trudinger- Moser result has been extended to sphere of higher order and Sobolev spaces over compact fields (see [7], [13]). Moreover, for any bounded Ω, the constant $c(\Omega, \propto)$ can be attained. For the attainability, we refer to[8], [5], [13] and (Li, 2001).

Another interesting extension of (1) is to construct Trudinger-Moser type inequalities on unbounded domains. When $n = 2$, this has been done in (Ruf, 2005). On the other hand, for unbounded domain in R^n.

Let

$$\Phi(t) = e^t - \sum_{j=1}^{n-2} \frac{t^i}{j|}$$

The result in (Li and Ruf, 2000) says that

Theorem C. For any $\propto \in (0, \propto_n)$ there is a constant $C(\propto)$ such that

$$\int_{R^n} \Phi\left(\propto \left(\frac{|u|}{\|\nabla u\|_{L^n(R^n)}} \right)^{\frac{n}{n-1}} \right) dx \le C(\propto) \frac{\|u\|_{L^n(R^n)}^n}{\|\nabla u\|_{L^n(R^n)}^n}, \quad for \quad u \in H^{1,n}(R^n) \setminus \{0\}. \qquad (2)$$

We shall discuss the critical case $\propto = \propto_n$. More precisely, we prove the following:

Theorem (1.1) (Adachi and Tanaka, 1999). There exists a constant $d > 1$, such that, for any domain $\Omega \subset R^n$,

$$\sup_{u \in H^{1,n}(\Omega), \|u\|_{H^{1,n}(\Omega)}} \int_\Omega \Phi\left(\propto_n |u|^{\frac{n}{n-1}} \right) dx \le d. \qquad (3)$$

The inequality is sharp: for any $\propto > \propto_n$, the supremum is $+\infty$.

We set

$$S = \sup_{u \in H^{1,n}(R^n), \|u\|_{H^{1,n}(R^n)}} \int_{R^n} \Phi\left(\propto_n |u|^{\frac{n}{n-1}} \right) dx.$$

Further, we will prove:

Theorem (1.2) (Ruf, 2005). S is attained. In other words, we can find a function $u \in H^{1,n}(R^n)$, WITH $\|u\|_{H^{1,n}(R^n)}$=1 such that

$$S = \int_{R^n} \Phi\left(\propto_n |u|^{\frac{n}{n-1}} \right) dx.$$

The second part of Theorem (1.2) is trivial. Given any fixed $\propto > \propto_n$, we take $\beta \in (\propto_n, \propto)$. By (1) we can find a positive sequence $\{ u_k \}$ in

$$\left\{ u \in H_0^{1.n}(B_1) : \int_{B_1} |\nabla u|^n dx = 1 \right\},$$

such that

$$\lim_{k \to +\infty} \int_{B_1} e^{\beta u_k^{\frac{n}{n-1}}} = +\infty.$$

By Lion's Lemma, we get $u_k \to 0$. Then by compact embedding theorem, we may assume $\|u_k\|_{L^p(B_1)} \to 0$ for any $p>1$. Then, $\int_{R^n}\left(|\nabla u_k|^n + |u_k|^n\right) dx \to 1$, and

$$\propto \left(\frac{u_k}{\|u_k\|_{H^{1,n}}}\right)^{\frac{n}{n-1}} > \beta u_k^{\frac{n}{n-1}}.$$

When k is sufficiently large. So, we get

$$\lim_{k\to+\infty}\int_{R^n}\Phi(\propto\left(\frac{u_k}{\|u_k\|_{H^{1,n}}}\right)^{\frac{n}{n-1}})\,dx \geq \lim_{k\to+\infty}\int_{B_1}(e^{\beta u_k^{\frac{n}{n-1}}}-1)\,dx = +\infty.$$

The first part of Theorem (1.1) and Theorem (1.2) will be proved by blow up analysis. We will use the ideas from [14] and (Li, 2005). However, in the unbounded case we do not obtain the strong convergence of u_k in $L^n(R^n)$, and so we have more techniques.

Concretely we will find positive and symmetric functions $u_k \in H^{1,n}(B_{R_k})$ which satisfy

$$\int_{B_{R_k}}\left(|\nabla u_k|^n + |u_k|^n\right)dx \to 1$$

and

$$\int_{B_{R_k}}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \sup_{\int_{B_{R_k}}\left(|\nabla v|^n+|v|^n\right)=1, v\in H_0^{1,n}(B_{R_k})}\int_{B_{R_k}}\Phi\left(\beta_k |v|^{\frac{n}{n-1}}\right)dx.$$

Here, β_k is an increasing sequence tending to $\propto_n$, and R_k is an increasing sequence tending to $+\infty$.

Further, u_k satisfies the following equation

$$-div|\nabla u_k|^{n-2}\nabla u_k + u_k^{n-1} = \frac{u_k^{\frac{1}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-2}}\right)}{\lambda_k},$$

where λ_k is Lagrange multiplier.

Then, there are two possibilities. If $c_k = \max u_k$ is bounded from above, then it is easy to see that

$$\lim_{k\to+\infty}\int_{R^n}\left(\Phi\left(\beta_k u_k^{\frac{n}{n-2}}\right) - \frac{\beta_k^{n-1}u_k^n}{(n-1)!}\right)dx = \int_{R^n}\left(\Phi\left(\propto_k u^{\frac{n}{n-2}}\right) - \frac{\propto_k^{n-1} u^n}{(n-1)!}\right)dx,$$

where u is the weak limit of $c_k = \max u_k$. It then follows that either $\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx$ converges to

$\int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx$ or $S \le \frac{\propto_n^{n-1}}{(n-1)!}$.

If c_k is not bounded, the key point of the proof is to show that

$$\frac{n}{n-1} \beta_k c_k^{\frac{1}{n-1}} \left(u_k(r_k x) - c_k\right) \to -n \log\left(1 + c_n^{\frac{n}{n-1}}\right),$$

locally for a suitably chosen sequence r_k (and with $c_n = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n-1}}$), and that

$$c_k^{\frac{1}{n-1}} u_k \to G,$$

On any $\Omega \subset\subset R^n \setminus \{0\}$, where G is some Green function.

In section (5.2), we will construct a function sequence $u_\in$ such that

$$\int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx > \frac{\omega_{n-1}}{n} e^{\propto_n A + 1 + 1/2 + \cdots + 1/(n-1)}$$

when $\in$ is sufficiently small. And also, we construct, for $n > 2$, a function sequence $u_\in$ such that for $\in$ sufficiently small

$$\int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx > \frac{\propto_n^{n-1}}{(n-1)!}.$$

Thus, together with Ruff's result of attainability in [14] for the case $n = 2$, we will get Theorem (1.2).

Definition (1.3) (Li and Ruf, 2000). To define the maximizing sequence, let $\{R_k\}$ be an increasing sequence which diverges to infinity, and $\{B_k\}$ an increasing sequence which converges to α_n. By compactness, we can find positive functions $u \in H^{1,n}(B_{R_k})$ with $\int_{B_{R_k}} \left(|\nabla u_k|^n + |u_k|^n\right) dx = 1$ such that

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \sup_{\int_{B_{R_k}} \left(|\nabla v|^n + |v|^n\right) = 1, v \in H_0^{1,n}(B_{R_k})} \int_{B_{R_k}} \Phi\left(\beta_k |v|^{\frac{n}{n-1}}\right) dx.$$

Moreover, we may assume $\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx$ is increasing.

Lemma (1.4) (Li and Ruf, 2000). Let u_k as above. Then

(a) u_k is a maximizing sequence for S;

(b) u_k may be chosen to be radially symmetric and decreasing.

Proof. (a) Let η be a cut-off function which is 1 on B_1 and 0 on $R^n \setminus B_2$. Then given any $\varphi \in H^{1,n}(R^n)$ with $\int_{R^n}\left(|\nabla\varphi|^n + |\varphi|^n\right)dx = 1$, we have

$$\tau^n(L) := \int_{R^n}\left(\left|\nabla\eta\left(\frac{x}{L}\right)\varphi\right|^n + \left|\eta\left(\frac{x}{L}\right)\varphi\right|^2\right)dx \to 1, \quad as \quad L \to +\infty.$$

Hence, for a fixed L and $R_k > 2L$

$$\int_{B_L} \Phi\left(\beta_k \left|\frac{\varphi}{\tau(L)}\right|^{\frac{n}{n-1}}\right)dx \le \int_{B_{2L}} \Phi\left(\beta_k \left|\frac{\eta\left(\frac{x}{L}\right)\varphi}{\tau(L)}\right|^{\frac{n}{n-1}}\right)dx \le \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \ .$$

By the Levi Lemma, we then have

$$\int_{B_L} \Phi\left(\propto_n \left|\frac{\varphi}{\tau(L)}\right|^{\frac{n}{n-1}}\right)dx \le \lim_{k\to+\infty}\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \ .$$

Then, letting $L \to +\infty$, we get

$$\int_{R^n} \Phi\left(\propto_n |\varphi|^{\frac{n}{n-1}}\right)dx \le \lim_{k\to+\infty}\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \ .$$

Hence, we get

$$\lim_{k\to+\infty}\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \sup_{\int_{R^n}\left(|\nabla v|^n + |v|^n\right)=1, v\in H_0^{1,n}\left(B_{R_k}\right)} \int_{R^n} \Phi\left(\propto_n |v|^{\frac{n}{n-1}}\right)dx.$$

(b) Let u_k^* be the radial rearrangement of u_k, then we have

$$\tau_k^n := \int_{B_{R_k}}\left(\left|\nabla u_k^*\right|^n + u_k^{*n}\right)dx \le \int_{B_{R_k}}\left(\left|\nabla u_k\right|^n + u_k^n\right)dx = 1.$$

It is well-known that $\tau_k = 1$ if and only if u_k is radial. Since

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{*\frac{n}{n-1}}\right)dx = \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \ ,$$

we have

$$\int_{B_{R_k}} \Phi\left(\beta_k \left(\frac{u_k^*}{\tau_k}\right)^{\frac{n}{n-1}}\right)dx \geq \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \ ,$$

And $"="$ holds if and only if $\tau_k = 1$. Hence $\tau_k = 1$ and

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{*\frac{n}{n-1}}\right)dx = \sup_{\int_{R^n}\left(|\nabla v|^n + |v|^n\right)=1, v\in H_0^{1,n}\left(B_{R_k}\right)} \int_{B_{R_k}} \Phi\left(\propto_n |v|^{\frac{n}{n-1}}\right)dx.$$

So, we can assume $u_k = u_k(|x|)$, and $u_k(r)$ is decreasing

Assume now $u_k = u$. Then, to prove Theorem (1.1) and Theorem (1.2), we only need to show that

$$\lim_{k\to+\infty} \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \int_{R^n} \Phi\left(\propto_n u_k^{\frac{n}{n-1}}\right)dx \ .$$

Definition (1.5) (Li and Ruf, 2000). By the definition of u_k, we have the equation

$$-div|\nabla u_k|^{n-2}\nabla u_k + u_k^{n-1} = \frac{u_k^{\frac{1}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)}{\lambda_k} \ , \qquad (4)$$

where λ_k is the constant satisfying

$$\lambda_k = \int_{B_{R_k}} u_k^{\frac{n}{n-2}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \ .$$

First, we need the following:

Lemma (1.6) (Li and Ruf, 2000). $\inf_k \lambda_k > 0$.

Proof. Assume $\lambda_k \to 0$. Then

$$\int_{R^n} u_k^n \, dx \leq C\int_{R^n} u_k^{\frac{n}{n-2}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \leq C\lambda_k \to 0 \, .$$

Since $u_k(|x|)$ is decreasing, we have $u_k^n(L)|B_L| \leq \int_{B_L} u_k^n \leq 1$, and then

$$u_k(L) \leq \frac{n}{\omega_n L^n} \ . \qquad (5)$$

set $\in = \frac{n}{\omega_n L^n}$. Then $u_k(x) \le \in$ for any $x \notin B_L$, and hence, we have, using the form of Φ, that

$$\lambda_k = \int_{R^n \setminus B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le C \int_{R^n \setminus B_L} u_k^n \, dx \le C\lambda_k \to 0 .$$

And on B_L, since $u_k \to 0$ in $L^q(B_L)$ for any $q > 1$, we have by Lebesgue

$$\lim_{k\to+\infty} \int_{B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \lim_{k\to+\infty} \left[\int_{B_L} C u_k^{\frac{n}{n-1}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + \int_{\{x \in B_L : u_k(x) \le 1\}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \right]$$

$$\le \lim_{k\to+\infty} C\lambda_k + \int_{B_L} \Phi(0) dx = 0$$

This is impossible. □

Results(1.7) (Shawgy and Mahgoub, 2011): (i) Definition (1.5) and results (2.8) implies that if we set $v_g = x$ and $\varphi(z) = z_e$ then we have

$$\Phi\left(B_k |u_k|^{\frac{n}{n-1}}\right) = e^{p_k |u_k|^{n/n-1}} .$$

(ii) Theorem (1.2) shows that

$$\propto \left(\frac{u_k}{\|u_k\|_{H^{1,n}}}\right)^{\frac{n}{n-1}} > \beta u_k^{\frac{n}{n-1}}.$$

$$\|u_k\|^{n/n-1} < \frac{\alpha^{\frac{n}{n-1}}}{\beta} \qquad \text{or} \qquad \|u_k\| < \frac{\alpha}{\beta^{n/n-1}} .$$

If $u_k \to u \in H^{1,n}(R^n)$ $with$ $\|u\|_{H^{1,n}(R^n)} = 1$ then $\beta < \alpha^{n/n-1}$.

(iii) If $c_k = Max\, u_k$, where u is the weak limit of $c_k = Max\, u_k$, it follows that

$$S \le \frac{\beta}{(n-1)1} \|u_k\| , \, n > 1 .$$

We denote $c_k = \max u_k = u_k(0)$. Then we have

Lemma (1.8) (Li and Ruf, 2000). If $\sup_k c_k < +\infty$,

(i) Theorem (5.1.1) holds;

(ii) if S is not attained, then

$S \leq \frac{\propto_n^{n-1}}{(n-1)!}$.

Proof. If $\sup_k c_k < +\infty$, then $u_k \to u$ in $C^1_{10c}(R^n)$. By (5), we are able to find L $s.t$ $u_k(x) \leq \in$ for $x \notin B_L$. Then

$$\int_{R^n \backslash B_L} \left(\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) - \frac{B_k^{n-1} u_k^n}{(n-1)!}\right) dx \leq C \int_{R^n \backslash B_L} u_k^n \, dx \leq C \in^{\frac{n^2}{n-1}-n} \int_{R^n} u_k^n \, dx \leq C \in^{\frac{n^2}{n-1}-n} .$$

Letting $\in \to 0$, we get

$$\int_{R^n \backslash B_L} \left(\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) - \frac{B_k^{n-1} u_k^n}{(n-1)|}\right) dx = \int_{R^n \backslash B_L} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n^{n-1} u^n}{(n-1)!}\right) dx .$$

Hence

$$\lim_{k \to +\infty} \int_{R^n \backslash B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \int_{R^n} \Phi\left(\propto_n u_k^{\frac{n}{n-1}}\right) dx + \frac{\propto_n^{n-1}}{(n-1)!} \lim_{k \to +\infty} \int_{R^n} \left(u_k^n - u^n\right) dx. \qquad (6)$$

When $u = 0$, we can denote from (6) that

$S \leq \frac{\propto_n^{n-1}}{(n-1)!}$.

Now, we assume $u \neq 0$. Set

$$\tau^n = \lim_{k \to +\infty} \frac{\int_{R^n} u_k^n dx}{\int_{R^n} u^n dx}.$$

By the Levi Lemma, we have $\tau \geq 1$.

Let $\bar{u} = u\left(\frac{x}{\tau}\right)$. Then, we have

$$\int_{R^n} |\nabla \bar{u}|^n dx = \int_{R^n} |\nabla u|^n dx \leq \lim_{k \to +\infty} \int_{R^n} |\nabla u_k|^n dx,$$

and

$$\int_{R^n} \bar{u}^n dx = \tau^n \int_{R^n} u^n dx = \lim_{k \to +\infty} \int_{R^n} u_k^n dx.$$

Then

$$\int_{R^n} \left(|\nabla \bar{u}|^n + \bar{u}^n\right) dx \leq \lim_{k \to +\infty} \int_{R^n} \left(|\nabla u_k|^n + |u_k^n|^n\right) dx = 1.$$

Hence, we have by (6)

$$S \geq \int_{R^n} \Phi\left(\propto_n \bar{u}^{\frac{n}{n-1}} \right) dx$$

$$= \tau^n \int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}} \right) dx$$

$$= [\int_{R^n} \Phi\left(\propto_n \bar{u}^{\frac{n}{n-1}} \right) dx + \left(\tau^n - 1\right)\int_{R^n} \frac{\propto_n^{n-1}}{(n-1)!} dx] + \left(\tau^n - 1\right)\int_{R^n} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n}{(n-1)!} \right) dx$$

$$= \lim_{k \to +\infty} \int_{R^n \setminus B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx + \left(\tau^n - 1\right)\int_{R^n} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n}{(n-1)|} \right) dx$$

$$= S + \left(\tau^n - 1\right)\int_{R^n} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n}{(n-1)!} \right) dx .$$

Since $\Phi\left(\propto_n' u^{\frac{n}{n-1}}\right) - \frac{\propto_n^{n-1}}{(n-1)!} u^n > 0$, we have $\tau = 1$, and then

$$S = \int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx .$$

So, u is an extremal function.

From now on, we assume $c_k \to +\infty$. We perform blow up procedure:

$$\tau_k^n = \frac{\lambda_k}{c_k^{\frac{n}{n-1}} e^{\beta_k c_k^{\frac{n}{n-1}}}} .$$

By (5) we can find a sufficiently L such that $u_k \leq 1$ on $R^n \setminus B_L$. Then

$$\int_{B_L} \left| \nabla\left(u_k - u_k(L)^+\right) \right|^n dx \leq 1,$$

and hence by (1), we have

$$\int_{B_L} e^{\propto_n \left|\left(u_k - u_k(L)^+\right)\right|^{\frac{n}{n-1}}} \leq C(L).$$

Clearly, for any $p < \propto_n$ we can find a constant $C(p)$, such that

$$p u_k^{\frac{n}{n-1}} \leq \propto_n \left| u_k - u_k(L)^+ \right|^{\frac{n}{n-1}} + C(p),$$

and then, we get

$$\int_{B_L} e^{pu_k^{\frac{n}{n-1}}} dx \le C = C(L, p).$$

Hence

$$\lambda_k e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}} = e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}} \left[\int_{R^n \setminus B_L} u_k^{\frac{n}{n-1}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + \int_{B_L} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \right]$$

$$\le C \int_{R^n \setminus B_L} u_k^n \, dx \; e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}} + \int_{B_L} e^{\frac{\beta_k}{2} u_k^{\frac{n}{n-1}}} u_k^{\frac{n}{n-1}} \, dx.$$

Since u_k converges in $L^q(B_L)$ for any $q > 1$, we get $\lambda_k \le C e^{\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}}$ and hence

$$r_k^n \le C e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}}.$$

Now, we set

$$v_k(x) = u_k(r_k x), \quad w_k(x) = \frac{n}{n-1} \beta_k C_k^{\frac{1}{n-1}} (v_k - c_k),$$

where v_k and w_k are defined on $\Omega_k = \{x \in R^n : r_k x \in B_1\}$. Using the definition of r_k^n and (4) we have

$$-div|\nabla w_k|^{n-2} \nabla w_k = \frac{v_k^{\frac{1}{n-1}}}{c_k^{\frac{1}{n-1}}} \left(\tfrac{n}{n-1} \beta_k\right)^{n-1} e^{\beta_k \left(v_k^{\frac{n}{n-1}} - c_k^{\frac{n}{n-1}}\right)} + O\left(r_k^n c_k^n\right).$$

In [9], we know that $osc_{B_R} w_k \le C(R)$ for any $R > 0$. Then from the result in (Dibendetto, 1983) (or [8]), it follows that $\|w_k\|_{C^{1,\delta}(B_R)} < C(R)$. Therefore w_k converges in C^1_{l0c} and $v_k - c_k \to 0$ in C^1_{l0c}.

Since

$$v_k^{\frac{n}{n-1}} = c_k^{\frac{n}{n-1}} \left(1 + \frac{v_k - c_k}{c_k}\right)^{\frac{n}{n-1}} = c_k^{\frac{n}{n-1}} \left(1 + \frac{n}{n-1} \frac{v_k - c_k}{c_k} + O\left(\frac{1}{c_k^2}\right)\right),$$

we get $\beta_k \left(v_k^{\frac{n}{n-1}} - c_k^{\frac{n}{n-1}}\right) \to w$ in C^1_{l0c}, and so we have

$$-div|\nabla w|^{n-2} \nabla w = \left(\frac{n \propto_n}{n-1}\right)^{n-1} e^w, \qquad (7)$$

with

$$w(0) = 0 = \max w.$$

Since w is radially symmetric and decreasing, it is easy to see that (7) has only one solution. We can check that

$$w(x) = -n\log\left(1 + c_n |x|^{\frac{n}{n-1}}\right), \quad \text{and} \quad \int_{R^n} e^w dx = 1,$$

where $c_n = \left(\frac{w_{n-1}}{n}\right)^{\frac{1}{n-1}}$. Then,

$$\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_{Lr_k}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx = \lim_{L\to+\infty} \int_{R^n} e^w dx = 1 . \qquad (8)$$

For $A > 1$, let $u_k^A = \min\{u_k, \frac{c_k}{A}\}$. We have

Lemma (1.9) (Li and Ruf, 2000). For any $A > 1$, there holds

$$\limsup_{k\to+\infty} \int_{R^n} \left(\left|\nabla u_k^A\right|^n + \left|u_k^A\right|^n \right) dx \le \frac{1}{A}. \qquad (9)$$

Proof. Since $\left|\left\{x : u_k \ge \frac{c_k}{A}\right\}\right| \left|\frac{c_k}{A}\right|^n \le \int_{\left\{u_k \ge \frac{c_k}{A}\right\}} u_k^n \le 1$, we can find a sequence $\rho_k \to 0$ such that

$$\{x : u_k \ge \frac{c_k}{A}\} \subset B_{\rho_k} .$$

Since u_k converges in $L^p(B_1)$ for any $p > 1$, we have

$$\lim_{k\to+\infty} \int_{\left\{u_k \ge \frac{c_k}{A}\right\}} \left|u_k^A\right|^p dx \le \lim_{k\to+\infty} \int_{\left\{u_k \ge \frac{c_k}{A}\right\}} u_k^p \, dx = 0 .$$

and

$$\lim_{k\to+\infty} \int_{R^n} \left(u_k - \frac{c_k}{A}\right)^+ u_k^p \, dx = 0$$

for any $p > 0$.

Hence, testing equation (4) with $\left(u_k - \frac{c_k}{A}\right)^+$, we have

$$\int_R \left(\left|\nabla\left(u_k - \frac{c_k}{A}\right)^+\right|^n + \left(u_k - \frac{c_k}{A}\right)^+ u_k^{n-1} \right) dx = \int_{R^n} \left(u_k - \frac{c_k}{A}\right)^+ \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx + o(1)$$

$$\ge \int_{B_{Lr_k}} \left(u_k - \frac{c_k}{A}\right)^+ \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx + o(1)$$

$$= \int_{B_L} \frac{v_k - \frac{c_k}{A}}{c_k} \left(\frac{v_k - \frac{c_k}{A}}{c_k} + 1\right)^{\frac{1}{n-1}} e^{w_k + o(1)} dx + o(1).$$

Hence

$$\liminf_{k\to+\infty}\int_R\left(\left|\nabla\left(u_k-\tfrac{c_k}{A}\right)^+\right|^n+\left(u_k-\tfrac{c_k}{A}\right)^+u_k^{n-1}\right)dx\ge\frac{A-1}{A}\int_{B_L}e^w dx.$$

Letting $L\to+\infty$, we get

$$\liminf_{k\to+\infty}\int_R\left(\left|\nabla\left(u_k-\tfrac{c_k}{A}\right)^+\right|^n+\left(u_k-\tfrac{c_k}{A}\right)^+u_k^{n-1}\right)dx\ge\frac{A-1}{A}.$$

Now, observe that

$$\int_{R^n}\left(\left|\nabla u_k^A\right|^n+\left|u_k^A\right|^n\right)dx= \quad 1-\int_R\left(\left|\nabla\left(u_k-\tfrac{c_k}{A}\right)^+\right|^n+\left(u_k-\tfrac{c_k}{A}\right)^+u_k^{n-1}\right)dx \quad +\int_{R^n}\left(u_k-\tfrac{c_k}{A}\right)^+dx- \quad \int_{\left\{u_k\ge\frac{c_k}{A}\right\}}u_k^n$$

$$\int_{\left\{u_k\ge\frac{c_k}{A}\right\}}\left|u_k^A\right|^n dx$$

$$\le 1-\left(1-\tfrac{1}{A}\right)+o(1).$$

Hence, we get this Lemma

Corollary (1.10) (Li and Ruf, 2000). We have

$$\int_{R^n\setminus B_\delta}\left(\left|\nabla u_k^A\right|+u_k^n\right)dx=0,$$

for any $\delta>0$, and then $u=0$.

Proof. Letting $A\to+\infty$, then for any constant c, we have

$$\int_{\{u_k\le c\}}\left(\left|\nabla u_k^A\right|+u_k^n\right)dx\to 0.$$

So, we get this Corollary.

Lemma (1.11) (Li and Ruf, 2000). We have

$$\lim_{k\to+\infty}\int_{R^n\setminus B_L}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \le \lim_{L\to+\infty}\lim_{k\to+\infty}\left(e^{\beta_k u_k^{\frac{n}{n-1}}}-1\right)dx=\limsup_{k\to+\infty}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}. \qquad (10)$$

and consequently

$$\frac{\lambda_k}{c_k}, \quad \text{and} \quad \sup_k\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}<+\infty. \qquad (11)$$

Proof. We have

$$\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \int_{\left\{u_k \le \frac{c_k}{A}\right\}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + \int_{\left\{u_k \le \frac{c_k}{A}\right\}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}}$$

$$\int_{R^n} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx .$$

Applying (5), we can find L such that $u_k \le 1$ on $R^n \setminus B_L$. Then by Corollary (1.11) and the form of Φ, we have

$$\lim_{k\to+\infty} \int_{R^n \setminus B_l} \Phi\left(p\beta_k (u_k^A)^{\frac{n}{n-1}}\right) dx \le \lim_{k\to+\infty} C(p) \int_{R^n \setminus B_l} u_k^n \, dx = 0 \qquad (12)$$

for any $p > 0$.

Since by Lemma (1.10) $\limsup_{k\to+\infty} \int_{R^n} \left(\left|\nabla u_k^A\right|^n + \left|u_k^A\right|^n \right) dx \le \frac{1}{A} < 1$, it follows from (1) that

$$\sup_k \int_{B_L} e^{p'\beta_k \left(u_k^A - u_k(L)^+\right)^{\frac{n}{n-1}}} dx < +\infty$$

for any $p' < A^{\frac{1}{n-1}}$. Since for any $p < p'$

$$p\left(u_k^A\right)^{\frac{n}{n-1}} \le p'\left(\left(u_k^A - u_k(L)^+\right)^{\frac{n}{n-1}} + C(p, p') \right),$$

we have

$$\sup_k \int_{B_L} \Phi\left(\beta_k \left(u_k^A\right)^{\frac{n}{n-1}}\right) dx < +\infty \qquad (13)$$

for any $p < A^{\frac{1}{n-1}}$. Then on B_L, by the weak compactness of Banach space, we get

$$\lim_{k\to+\infty} \int_{B_L} \Phi\left(\beta_k \left(u_k^A\right)^{\frac{n}{n-1}} \right) dx = \int_{B_L} \Phi(0) dx = 0 .$$

Hence, we have

$$\lim_{k\to+\infty} \int_{B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \lim_{L\to+\infty} \lim_{k\to+\infty} A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}} \int_{B_L} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + C_\in = \lim_{K\to+\infty} A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}} + C_\in .$$

As $A \to 1$ and $\in \to 0$, we obtain (10).

If $\frac{\lambda_k}{c_k}$ was bounded or $\sup_k \frac{c_k^{\frac{n}{n-1}}}{\lambda_k} = +\infty$, it would follow from (10) that

$$\sup_{\int_{R^n}\left(|\nabla v|^n+|v|^n\right)=1, v\in H_0^{1,n}\left(B_{R_k}\right)} \int_{B_{R_k}} \Phi\left(\propto_n |v|^{\frac{n}{n-1}}\right)dx = 0.$$

Which is impossible.

Lemma (1.12) (Carleson and Chang, 1986). We have that $c_k \frac{u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)$ converges to δ_0 weakly, *i.e.* for any $\varphi \in D\left(R^n\right)$

$$\lim_{k\to+\infty} \varphi c_k \frac{u_k^{\frac{1}{n-1}}}{\lambda_k}\int_{B_L} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \varphi(0).$$

Proof. Suppose $supp\,\varphi \subset B_p$. We split the integral

$$\int_{B_p} \varphi \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \le \int_{\left\{u_k \ge \frac{c_k}{\lambda}\right\}\setminus B_{Lr_k}} \cdots + \int_{B_{Lr_k}} \cdots + \int_{\left\{u_k < \frac{c_k}{\lambda}\right\}} \cdots = I_1 + I_2 + I_3.$$

We have

$$I_1 \le A\|\varphi\|_{C^0}\int_{R^n\setminus B_{Lr_k}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = A\|\varphi\|_{C^0}\left(1-\int_{B_L} e^{w_k+o(1)}dx\right),$$

and

$$I_2 = \int_{B_L}\varphi(r_k x)\frac{c_k\left(c_k+\left(v_k-c_k\right)\right)^{\frac{1}{n-1}}}{c_k^{\frac{n}{n-1}}}e^{w_k+o(1)}dx = \varphi(0)\int_{B_L} e^w dx + o(1) = \varphi(0)+o(1).$$

By (12) and (13) we have

$$\int_{R^n}\Phi\left(p\beta_k\,|u_k^A|^{\frac{n}{n-1}}\right)dx < C$$

for any $p < A^{\frac{1}{n-1}}$. We set $\frac{1}{q}+\frac{1}{p}=1$. Then we get by (11)

$$I_3 = \int_{\left\{u_k \le \frac{c_k}{\lambda}\right\}} \varphi c_k \frac{u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \le \frac{c_k}{\lambda_k}\|\varphi\|_{C^0}\left\|u_k^{\frac{1}{n-1}}\right\|_{L^q\left(R^n\right)}\left\|e^{\beta_k\left|u_k^A\right|^{\frac{n}{n-1}}}\right\|_{L^q\left(R^n\right)} \to 0.$$

Letting $L \to +\infty$, we deduce now that

$$\lim_{k\to+\infty}\int_{R^n}\varphi\frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \varphi(0)$$

Proposition (1.13) (Li and Ruf, 2000). On any $\Omega \subset\subset R^n \setminus \{0\}$, we have that $c_k^{\frac{1}{n-1}} u_k$ converges to G in $C'(\Omega)$, where $G \in C_{loc}^{1,\propto}\left(R^n \setminus \{0\}\right)$ satisfies the following equation

$$-div|\nabla G|^{n-2}\nabla G + G^{n-1} = \delta_0 . \qquad (14)$$

Proof. We set $U_k = c_k^{\frac{1}{n-1}} u_k$, which satisfy by (4) the equations:

$$-div|\nabla U_k|^{n-2}\nabla U + U_k^{n-1} = \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right). \qquad (15)$$

For our purpose, we need to prove

$$\int_{B_R} |U_k|^q dx \le C(q,R),$$

where $C(q,R)$ does not depend on k. We use the idea in [80] to prove this statement.

Set $\Omega_t = \{0 \le U_k \le t\}, U_k^t = \min\{U_k, t\}$. Then we have

$$\int_{\Omega_t}\left(|\nabla U_k|^n + |U_k|^n\right)dx \le \int_{R^n}\left(-U_k^t \Delta_n U_k + U_k^t U_k^{n-1}\right) = \int_{R^n} U_k^t \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) \le 2t .$$

Let η be a radially symmetric cut-off function which is 1 on B_R and 0 on B_{2R}^c. Then,

$$\int_{B_{2R}} \left|\nabla \eta U_k^t\right|^n dx \le C_1(R) + C_2(R)t .$$

Then, when t is bigger than $\frac{C_1(R)}{C_2(R)}$, we have

$$\int_{B_{2R}} \left|\nabla \eta U_k^t\right|^n dx \le 2C_2(R)t .$$

Set ρ such that $U_k(\rho) = t$. Then we have

$$\inf\{\int_{B_{2R}} |\nabla v|^n dx : v \in H_0^{1,n}(B_{2R}) \quad \text{and} \quad v|_{B_\rho} = t\} \le 2C_2(R)t.$$

On the other hand, the $\inf$ is achieved by $-t\log\frac{|x|}{2R} \Big/ \log\frac{2R}{\rho}$. By a direct computation, we have

$$\frac{\omega_{n-1} t^{n-1}}{\left(\log\frac{2R}{\rho}\right)^{n-1}} \le 2R ,$$

and hence for any $t > \frac{C_1(R)}{C_2(R)}$

$$\left|\{x \in B_{2R} : U_k \geq t\}\right| = \left|B_\rho\right| \leq C_3(R)e^{-A(R)t},$$

where $A(R)$ is a constant only depending on R. Then, for any $\delta < A$,

$$\int_{B_R} e^{\delta U_k} dx \leq \sum_{m=0}^{\infty} \mu(\{m \leq U_k \leq m+1\}) e^{\delta(m+1)} \leq \sum_{m=0}^{\infty} e^{-(A-\delta)m} e^{\delta} \leq C.$$

Then, testing the equation (15) with the function $\log\frac{1+2(U_k - U_k(R))^+}{1-(U_k - U_k(R))^+}$, we get

$$\int_{B_R} \frac{|\nabla U_k|^n}{(1+U_k - U_k(R))(1+2U_k - 2U_k(R))} dx$$

$$\leq \log 2\int_{B_R} \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx - \int_{B_R} U_k \log\frac{1+2(U_k - U_k(R))^+}{1-(U_k - U_k(R))^+} dx \leq C.$$

Given $q < n$, by Young's inequality, we have

$$\int_{B_R} |\nabla U_k|^q dx \leq \int_{B_R} \left[\frac{|\nabla U_k|^n}{(1+U_k - U_k(R))(1+2U_k - 2U_k(R))} dx + ((1+U_k)(1+2U_k))^{\frac{n}{n-1}} \right]$$

$$\leq \int_{B_R} \left[\frac{|\nabla U_k|^n}{(1+U_k - U_k(R))(1+2U_k - 2U_k(R))} dx + Ce^{\delta U_k} \right] dx.$$

Hence, we are able to assume that U_k converges to a function G weakly in $H^{1,p}(B_R)$ for any R and $p < n$. Applying Lemma (1.12), we get (14).

Hence U_k is bounded in $L^q(\Omega)$ for any $q > 0$. By Corollary (1.10) and Theorem C, $e^{\beta_k u_k^{\frac{n}{n-1}}}$ is also bounded in $L^q(\Omega)$ for any $q > 0$. Then applying [9], and [8](or [5]), we get $\|U_k\|_{C^{1,\propto}(\Omega)} \leq C$. So U_k converges to G in $C^1(\Omega)$.

For the Green function G we have the following result.

Lemma (1.14) (Li and Ruf, 2000). $G \in C^{1,\propto}_{loc}(R^n \setminus \{0\})$ and near 0 we can write

$$G = -\frac{1}{\propto_n} \log r^n + A + O(r^n \log^n r);$$

here, A is a constant. Moreover, for any $\delta > 0$, we have

$$\lim_{k\to+\infty}\int_{R^n\setminus B_\delta}\left(\left|\nabla c_k^{\frac{i}{n-1}}u_k\right|^n+\left(c_k^{\frac{1}{n-1}}u_k\right)^n\right)dx=\int_{R^n\setminus B_\delta}\left(|\nabla G|^n+(G)^n\right)dx$$

$$=G(\delta)\left(t-\int_{B_\delta}G^{n-1}dx\right).$$

Proof. Slightly modifying the proof in [14], we can prove

$$G=-\frac{1}{\propto_n}\log r^n+A+o(1).$$

One can see [26] for details. Further, testing the equation (15) with 1, we get

$$\omega_{n-1}G^d(r)^{n-1}r^{n-1}=\int_{\partial B}|\nabla G|^{n-2}\frac{\partial G}{\partial n}=1-\int_{B_r}G^{n-1}dx=1+O\left(r^n\log^{n-1}r\right).$$

Then we get (16).

We have

$$\int_{R^n\setminus B_\delta}u_k^{\frac{n}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx\le C\int_{R^n\setminus B_\delta}u_k^n\,dx\to 0. \qquad (17)$$

Recall that $U_k\in H_0^{1,n}\left(B_{R_k}\right)$. By equation (15) we get

$$\int_{R^n\setminus B_\delta}\left(|\nabla U_k|^n+U_k^n\right)dx=\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}\int_{R^n\setminus B_\delta}u_k^{\frac{n}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx-\int_{\partial B_\delta}\frac{\partial U_k}{\partial n}|\nabla U_k|^{n-2}U_k dS.$$

By (17) and (11) we then get

$$\lim_{k\to+\infty}\int_{R^n\setminus B_\delta}\left(|\nabla U_k|^n+U_k^n\right)dx=\lim_{k\to+\infty}\int_{\partial B_\delta}\frac{\partial U_k}{\partial n}|\nabla U_k|^{n-2}U_k dS$$

$$=-G(\delta)\int_{\partial B_\delta}\frac{\partial G}{\partial n}|\nabla G|^{n-2}dS$$

$$=G(\delta)\left(1-\int_{B_\delta}G^{n-1}dx\right).$$

We are now in the position to complete the proof of Theorem (1.1): We have seen in (12) that

$$\int_{R^n\setminus B_\delta}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx\le C.$$

So, we only need to prove on B_R

$$\int_{B_R}e^{\beta_k u_k^{\frac{n}{n-1}}}\,dx<C.$$

The classical Trudinger-Moser inequality implies that

$$\int_{B_R} e^{\beta_k \left(u_k - u_k(R)^+\right)^{\frac{n}{n-1}}} dx < C = C(R).$$

By Proposition (1.14), $u_k(R) = O\left(\frac{1}{c_k^{\frac{1}{n-1}}}\right)$, and hence we have

$$u_k^{\frac{n}{n-1}} \le \left((u_k - u_k(R))^+ + u_k(R)\right)^{\frac{n}{n-1}} \le \left((u_k - u_k(R))^+\right)^{\frac{n}{n-1}} + C_1,$$

then, we get

$$\int_{B_R} e^{\beta_k u_k^{\frac{n}{n-1}}} dx < C'.$$

To proof Proposition (1.16), we will use a result of Carleson and Chang (see [12]:

Lemma (1.15) (Li and Ruf, 2000). Let B be the unit ball in R^n. Assume that u_k is a sequence in $H_0^{1,n}(B)$ with $\int_B |\nabla U_k|^n dx = 1$. If $u_k \to 0$, then

$$\limsup_{k\to+\infty} \int_B \left(e^{\propto_n |u_k|^{\frac{n}{n-1}}} - 1\right) dx \le |B| e^{1+1/2+\cdots+1/(n-1)}.$$

Then, we get the following:

Proposition (1.16) (Li and Ruf, 2000). If S cannot be attained, then

$$S > \min\left\{\frac{\propto_n^{n-1}}{(n-1)!} e^{\propto_n A + 1 + 1/2 + \cdots + 1/(n-1)}\right\}.$$

Proof. Set $u_k' = \frac{(u_k(x) - u_k(\delta))^+}{\|\nabla u_k\|_{L^n(B_\delta)}}$ which is in $H_0^{1,n}(B_\delta)$. Then by the result of Carleson and Chang, we have

$$\limsup_{k\to+\infty} \int_{B_\delta} e^{\beta_k u_k'^{\frac{n}{n-1}}} \le |B_\delta| e^{1+1/2+\cdots+1/(n-1)}.$$

By Lemma (1.15), We have

$$\int_{R^n \setminus B_\delta} \left(\left|\nabla c_k^{\frac{1}{n-1}} u_k\right|^n + \left(c_k^{\frac{1}{n-1}} u_k\right)^n\right) dx \to G(\delta)\left(1 - \int_{B_\delta} G^{n-1} dx\right),$$

and therefore, we get

$$\int_{B_\delta} |\nabla u_k|^n dx = 1 - \int_{R^n \setminus B_\delta} \left(|\nabla u_k|^n + u_k^n\right) dx - \int_{B_\delta} u_k^n dx = 1 - \frac{G(\delta) - \in_k(\delta)}{c_k^{\frac{n}{n-1}}}, \quad (18)$$

where $\lim_{\delta\to 0}\lim_{k\to+\infty} \in_k = 0$.

By (12) in Lemma (1.11) we have

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_\rho\setminus B_{Lr_k}} e^{\beta_k u_k'^{\frac{n}{n-1}}} = \left|B_\rho\right|,$$

for any $\rho<\delta$. Furthermore, on B_ρ we have by (18)

$$\left(u_k'\right)^{\frac{n}{n-1}} \le \frac{u_k^{\frac{n}{n-1}}}{\left(1-\frac{G(\delta)+\in_k(\delta)}{c_k^{\frac{n}{n-1}}}\right)^{\frac{1}{n-1}}} = u_k^{\frac{n}{n-1}}\left(1+\frac{1}{n-1}\frac{G(\delta)+\in_k(\delta)}{c_k^{\frac{n}{n-1}}}+O\left(\frac{1}{c_k^{\frac{2n}{n-1}}}\right)\right)$$

$$= u_k^{\frac{n}{n-1}} + \frac{1}{n-1}G(\delta)\left(\frac{u_k}{c_k}\right) + O\left(c_k^{\frac{-n}{n-1}}\right)$$

$$\le u_k^{\frac{n}{n-1}} - \frac{\log\delta^2}{(n-1)\propto_n}.$$

Then we have

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_\rho\setminus B_{Lr_k}} e^{\beta_k u_k'^{\frac{n}{n-1}}}dx \le O\left(\delta^{-n}\right)\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_\rho\setminus B_{Lr_k}} e^{\beta_k u_k^{\frac{n}{n-1}}}dx \to \left|B_\rho\right|O\left(\delta^{-n}\right).$$

since $u_k' \to 0$ on $B_\delta\setminus B_\rho$, we get

$$\lim_{k\to+\infty}\int_{B_\delta\setminus B_\rho}\left(e^{\beta_k u_k'^{\frac{n}{n-1}}}-1\right)dx = 0,$$

then

$$0 \le \lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_\delta\setminus B_{Lr_k}}\left(e^{\beta_k u_k'^{\frac{n}{n-1}}}-1\right)dx \le \left|B_\rho\right|O\left(\delta^{-n}\right).$$

Letting $\rho\to 0$, we get

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_\delta\setminus B_{Lr_k}}\left(e^{\beta_k u_k'^{\frac{n}{n-1}}}-1\right)dx = 0.$$

So we have

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{B_\delta\setminus B_{Lr_k}}\left(e^{\beta_k u_k'^{\frac{n}{n-1}}}-1\right)dx \le e^{1+1/2+\cdots+1/(n-1)}\left|B_\delta\right|.$$

Now, we fix an L. Then for any $x\in B_{Lr}$, we have

$$\beta_k u_k^{\frac{n}{n-1}} = \beta_k\left(\frac{u_k}{\left\|\nabla u_k\right\|_{L^n(B_\delta)}}\right)^{\frac{n}{n-1}}\left(\int_{B_\delta}\left|\nabla u_k\right|^n dx\right)^{\frac{1}{n-1}}$$

$$= \beta_k \big(u'_k + \frac{u_k(\delta)}{\|\nabla u_k\|_{L^n(B_\delta)}}\big)^{\frac{n}{n-1}} \big(\int_{B_\delta} |\nabla u_k|^n \, dx\big)^{\frac{1}{n-1}}$$

(using that $u_k(\delta) = O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)$ and $\|\nabla u_k\|_{L^n(B_\delta)} = 1 + O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)$)

$$= \beta_k \big(u' +_k u_k(\delta) + O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)\big)^{\frac{n}{n-1}} \big(\int_{B_\delta} |\nabla u_k|^n \, dx\big)^{\frac{1}{n-1}}$$

$$= \beta_k u_k'^{\frac{n}{n-1}} \big(1 + \frac{u_k(\delta)}{u'_k} + O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)\big)^{\frac{n}{n-1}} \left(1 - \frac{G(\delta) + \in_k(\delta)}{c_k^{\frac{n}{n-1}}}\right)^{\frac{1}{n-1}}$$

$$= \beta_k u_k'^{\frac{n}{n-1}} \left[1 + \frac{n}{n-1} \frac{u_k(\delta)}{u'_k} - \frac{1}{n-1} \frac{G(\delta) + \in_k(\delta)}{c_k^{\frac{n}{n-1}}} + O\left(\frac{1}{c_k^{\frac{2n}{n-1}}}\right)\right].$$

It is easy to check that

$$\frac{u'_k(r_k x)}{c_k} \to 1, \quad \text{and} \quad (u'_k(r_k x))^{\frac{1}{n-1}} u_k(\delta) \to G(\delta).$$

So, we get

$$\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_{Lr_k}} \big(e^{\beta_k u_k^{\frac{n}{n-1}}} - 1\big) dx = \lim_{L\to+\infty} \lim_{k\to+\infty} e^{\propto_n G(\delta)} \int_{B_{Lr_k}} \big(e^{\beta_k u_k'^{\frac{n}{n-1}}} - 1\big) dx$$

$$\le e^{\propto_n G(\delta)} \delta^n \frac{\omega_{n-1}}{n} \le e^{1+1/2+\cdots+1/(n-1)}$$

$$= e^{\propto_n \left(-\frac{1}{\propto_n} \log \delta^n + A + O(\delta^n \log^n \delta)\right)} \delta^n \frac{\omega_{n-1}}{n} \, e^{1+1/2+\cdots+1/(n-1)}.$$

letting $\delta \to 0$, then the above inequality together with Lemma (1.8) imply Proposition (1.16).

1. The test functions

Definition (2.1) (Li and Ruf, 2000). We will construct a function sequence $\{u_\in\} \subset H^{1,n}(R^n)$ with $\|u_\in\|_{H^{1,n}} = 1$ which satisfies

$$\int_{R^n} \Phi\Big(\propto_n |u_\in|^{\frac{n}{n-1}}\Big) dx > \frac{\omega_{n-1}}{n} e^{A+1+1/2+\cdots+1/(n-1)},$$

for $\in > 0$ sufficiently small.

Let

$$u_\in = \begin{cases} C - \dfrac{(n-1)\log\left(1 + c_n \left|\frac{x}{\in}\right|^{\frac{n}{n-1}}\right) + A_\in}{\propto_n C^{\frac{1}{n-1}}} & |x| \le L_\in \\ \dfrac{G(|x|)}{C^{\frac{1}{n-1}}} & |x| > L_\in . \end{cases}$$

where $A_\in, C$ and L are functions of $\in$ (which will be defined later, by (19), (20), (21)) which satisfy

$(i)\, L \to +\infty, C \to +\infty$ and $L_\in \to 0$, as $\in \to 0$;

$(ii)\, C - \dfrac{(n-1)\log\left(1 + c_n L^{\frac{n}{n-1}}\right) + A_\in}{\propto_n C^{\frac{1}{n-1}}}$

$(iii)\, \dfrac{\log L}{C^{\frac{n}{n-1}}} \to 0 \quad as \quad \in \to 0.$

We use the normalization of $u_\in$ to obtain information on $A_\in, C$ and L. we have

$$\int_{R^n \setminus B_{L_\in}} \left(|\nabla u_\in|^n + u_\in^n\right) d\,x = \frac{1}{C^{\frac{n}{n-1}}}\left(\int_{B_{L_\in}^\in} |\nabla G|^n\, dx + \int_{B_{L_\in}^\in} G^n dx\right)$$

$$= \frac{1}{C^{\frac{n}{n-1}}} \int_{\partial B_{L_\in}} G(L_\in) |\nabla G|^{n-2} \frac{\partial G}{\partial n} dS$$

$$= \frac{G(L_\in) - G(L_\in) \int_{B_{L_\in}} G dx}{C^{\frac{n}{n-1}}},$$

And

$$\int_{B_{L_\in}} |\nabla u_\in|^n\, dx = \frac{n-1}{\propto_n C^{\frac{n}{n-1}}} \int_0^{c_n L^{\frac{n}{n-1}}} \frac{u^{n-1}}{(1+u)^n} du$$

$$= \frac{n-1}{\propto_n C^{\frac{n}{n-1}}} \int_0^{c_n L^{\frac{n}{n-1}}} \frac{((1+u)-1)^{n-1}}{(1+u)^n} du$$

$$= \frac{n-1}{\propto_n C^{\frac{n}{n-1}}} \sum_{k=0}^{n-2} \frac{C_{n-1}^k (-1)^{n-1-k}}{n-1-k}$$

$$= \frac{n-1}{\propto_n C^{\frac{n}{n-1}}} \log\left(1 + c_n L^{\frac{n}{n-1}}\right) + O\left(\frac{1}{L^{\frac{n}{n-1}} C^{\frac{n}{n-1}}}\right)$$

$$= -\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\left(1+1/2+1/3+\cdots+1/(n-1)\right)+\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\log\left(1+c_n L^{\frac{n}{n-1}}\right)+O\left(\frac{1}{L^{\frac{n}{n-1}}C^{\frac{n}{n-1}}}\right),$$

where we used the fact

$$-\sum_{k=0}^{n-2}\frac{C_{n-1}^{k}(-1)^{n-1-k}}{n-1-k}=1+\frac{1}{2}+\cdots+\frac{1}{n-1}.$$

It is easy to check that

$$\int_{B_{L_\in}}|\nabla u_\in|^n\,dx=O\left((L_\in)^n C^n\log L\right),$$

and thus we get

$$\int_{R^n}\left(|\nabla u_\in|^n+u_\in^n\right)dx=\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\{-(n-1)\left(1+1/2+1/3+\cdots+1/(n-1)\right)+\propto_n$$

$$+(n-1)\log\left(1+c_n L^{\frac{n}{n-1}}\right)-\log(L_\in)+\phi\},$$

where

$$\phi=O\left((L_\in)^n C^n\log L+(L_\in)^n\log^n L_\in+L^{\frac{n}{n-1}}\right).$$

Setting $\int_{R^n}\left(|\nabla u_\in|^n+u_\in^n\right)dx=1$, we obtain

$$\propto_n C^{\frac{n}{n-1}}=-(n-1)\left(1+1/2+\cdots+1/(n-1)\right)+\propto_n A+\log\frac{\left(1+c_n L^{\frac{n}{n-1}}\right)^{n-1}}{L^n}-\log\in^n+\phi$$

$$=-(n-1)\left(1+1/2+\cdots+1/(n-1)\right)+\propto_n A+\log\frac{\omega_{n-1}}{n}-\log\in^n+\phi. \qquad (19)$$

By (ii) we have

$$\propto_n C^{\frac{n}{n-1}}-(n-1)\log\left(1+c_n L^{\frac{n}{n-1}}\right)+A_\in=\propto G(L_\in)$$

and hence

$$-(n-1)\left(1+1/2+\cdots+1/(n-1)\right)+\propto_n A-\log(L_\in)^n+\phi+A_\in=\propto G(L_\in);$$

this implies that

$$A_{\notin}=-(n-1)\left(1+1/2+\cdots+1/(n-1)\right)+\phi. \qquad (20)$$

Next, we compute $\int_{B_{L_\in}}e^{\propto_n|u|_n^{\frac{n}{n-1}}}dx$.

Clearly, $\varphi(t)=|1-t|^{\frac{n}{n-1}}+\frac{n}{n-1}t$ is increasing when $0\le t\le 1$ and decreasing when $t\le 0$, then

$|1-t|^{\frac{n}{n-1}}\ge 1-\frac{n}{n-1}t$, when $|t|<1$.

Thus we have by (ii), for any $x\in B_{L_\in}$

$$\propto_n u_\in^{\frac{n}{n-1}} = \propto_n C^{\frac{n}{n-1}}\left|1-\frac{(n-1)\log\left(1+c_n\left|\frac{x}{\in}\right|^{\frac{n}{n-1}}\right)+A_\in}{\propto_n C^{\frac{n}{n-1}}}\right|$$

$$\ge \propto_n C^{\frac{n}{n-1}}\left(1-\frac{n}{n-1}\frac{(n-1)\log\left(1+c_n\left|\frac{x}{\in}\right|+A_\in\right)}{\propto_n C^{\frac{1}{n-1}}}\right). \qquad (21)$$

Then we have

$$\int_{B_{L_\in}} e^{\propto_n |u_\in|^{\frac{n}{n-1}}_n}dx \ge \int_{B_{L_\in}} e^{\propto_n C^{\frac{n}{n-1}-n\log\left(1+c_n\left|\frac{x}{\in}\right|^{\frac{n}{n-1}}\right)-\frac{n}{n-1}A_\in}}dx$$

$$= e^{\propto_n C^{\frac{n}{n-1}}-\frac{n}{n-1}A_\in}\int_B \frac{\in^n}{\left(1+c_n|x|^{\frac{n}{n-1}}\right)^n}dx$$

$$= e^{\propto_n C^{\frac{n}{n-1}}-\frac{n}{n-1}A_\in}\int_0^{c_n L^{\frac{n}{n-1}}}\frac{u^{n-1}}{(1+u)^n}du$$

$$= e^{\propto_n C^{\frac{n}{n-1}}-\frac{n}{n-1}A_\in}(n-1)\int_0^{c_n L^{\frac{n}{n-1}}}\frac{((u+1)-1)^{n-2}}{(1+u)^n}du$$

$$= e^{\propto_n C^{\frac{n}{n-1}}-\frac{n}{n-1}A_\in}\in^n\left(1+O\left(L^{-\frac{n}{n-1}}\right)\right)$$

$$= \frac{\omega_{n-1}}{n}e^{A+1+1/2+\cdots+1/(n-1)}+O\left((L_\in)^n C^n\log L+(L_\in)^n\log^n L_\in+L^{-\frac{n}{n-1}}\right).$$

Here, we used the fact

$$\sum_{k=0}^{m}\frac{(-1)^{m-k}}{m-k-1}C_m^k=\frac{1}{m+1}.$$

Then

$$\int_{B_{L_\in}}\Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right)dx \ge \frac{\omega_{n-1}}{n}e^{\propto_n A+1+1/2+\cdots+1/(n-1)}+O\left((L_\in)^n C^n\log L+(L_\in)^n\log^n L_\in+L^{-\frac{n}{n-1}}\right).$$

Moreover, on $R^n\setminus B_{L_\in}$ we have the estimate

$$\int_{R^n \setminus B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right) dx \geq \frac{\propto_n^{n-1}}{(n-1)!} \int_{R^n \setminus B_{L_\in}} \left| \frac{G(x)}{C^{\frac{1}{n-1}}} \right|^n dx,$$

and thus, we get

$$\int_{B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right) dx \geq \frac{\omega_{n-1}}{n} e^{\propto_n A + 1 + 1/2 + \cdots + 1/(n-1)} + \frac{\propto_n^{n-1}}{(n-1)!} \int_{R^n \setminus B_{L_\in}} \left| \frac{G(x)}{C^{\frac{1}{n-1}}} \right|^n dx$$

$$+ O\left((L_\in)^n C^n \log L + (L_\in)^n \log^n L_\in + L^{-\frac{n}{n-1}} \right)$$

$$= \frac{\omega_{n-1}}{n} e^{\propto_n A + 1 + 1/2 + \cdots + 1/(n-1)} + \frac{\propto_n^{n-1}}{(n-1)! C^{\frac{n}{n-1}}} \left[\int_{R^n \setminus B_{L_\in}} |G(x)|^n dx \right.$$

$$\left. + O\left((L_\in)^n C^{n+\frac{n}{n-1}} \log L + \frac{C^{\frac{n}{n-1}}}{L^{\frac{n}{n-1}}} + C^{\frac{n}{n-1}} (L_\in)^n \log^n L_\in \right) \right] \quad (22)$$

We now set

$$L = -\log \in; \quad (23)$$

then $L_\in \to 0$ as $\in \to 0$. We then need to prove that there exists $C = C(\in)$ which solves equation (19). We set

$$f(t) = -\propto_n t^{\frac{n}{n-1}} - (n-1)(1 + 1/2 + \cdots + 1/(n-1)) + \propto_n A + \log \frac{\omega_{n-1}}{n} - \log \in^n + \phi,$$

since

$$f\left(\left(-\tfrac{2}{\propto_n} \log \in^n\right)^{\frac{n}{n-1}}\right) = \log \in^n + o(1) + \phi < 0$$

for $\in$ small, and

$$f\left(\left(-\tfrac{1}{2\propto_n} \log \in^n\right)^{\frac{n}{n-1}}\right) = -\frac{1}{2} \log \in^n + o(1) + \phi > 0$$

for $\in$ small, *f* has a zero in $f\left(\left(-\tfrac{1}{2\propto_n} \log \in^n\right)^{\frac{n}{n-1}}\right)$, $f\left(\left(-\tfrac{2}{\propto_n} \log \in^n\right)^{\frac{n}{n-1}}\right)$. Thus, we defined *C*, and it satisfies

$$\propto_n C^{\frac{n}{n-1}} = -\log \in^n + O(1).$$

Therefore, as $\in \to 0$, we have

$$\frac{\log L}{C^{\frac{n}{n-1}}} \to 0,$$

and then

$$(L_\in)^n C^{n+\frac{n}{n-1}} \log L + C^{\frac{n}{n-1}} L^{\frac{-n}{n-1}} + C^{\frac{n}{n-1}} (L_\in)^n \log^n L_\in \to 0.$$

Therefore, $(i),(ii),(iii)$ hold and we can conclude from (22) that for $\in > 0$ sufficiently small

$$\int_{B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right)dx > \frac{\omega_{n-1}}{n} e^{\propto_n A+1+1/2+\cdots+1/(n-1)}.$$

Definition (2.2) (Li and Ruf, 2000). To define the test function 2, we construct, for $n > 2$, functions $u_\in$ such that

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_\in}{\|u_\in\|_{H^{1,n}}}\right)^{\frac{n}{n-1}}\right)dx > \frac{\propto_n^{n-1}}{(n-1)!},$$

for $\in > 0$ sufficiently small.

Let $\in^n = e^{-\propto_n c^{\frac{n}{n-1}}}$, and

$$u_\in = \begin{cases} c & |x| < L_\in \\ \dfrac{-n\log\frac{x}{L}}{\propto_n c^{\frac{1}{n-1}}} & L_\in \le |x| \le L \\ 0 & L \le |x|; \end{cases}$$

where L is a function of $\in$ which will be defined later.

We have

$$\int_{R^n} |\nabla u_\in|^n = 1,$$

and

$$\int_{R^n} u_\in^n dx = \frac{\omega_{n-1}}{n} c^n (L_\in)^n + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_\in^1 r^{n-1} \log^n r \, dr.$$

Then

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_\in}{\|u_\in\|_{H^{1,n}}}\right)^{\frac{n}{n-1}}\right)dx \ge \frac{\propto_n^{n-1}}{(n-1)!} \frac{\int_{R^n} u_\in^n dx}{1+\int_{R^n} u_\in^n dx} + \frac{\propto_n^{n-1}}{n!} \frac{\int_{R^n \setminus B_{L_\in}} u_\in^{\frac{n^2}{n-1}}}{\left(1+\int_{R^n} u_\in^n dx\right)^{\frac{n}{n-1}}} dx$$

$$= \frac{\propto_n^{n-1}}{(n-1)!} - \frac{\propto_n^{n-1}}{(n-1)!} \frac{1}{1+\frac{\omega_{n-1}}{n} c^n (L_\in) + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_\in^1 r^{n-1} \log^n r \, dx}$$

$$=\frac{\propto^n}{n!}\frac{\omega_{n-1}L^n / c^{\frac{n^2}{(n-1)^2}}\int_{\in}^{1} r^{n-1}\log^{\frac{n^2}{n-1}} r}{\left(1+\frac{\omega_{n-1}}{n}c^n\left(L_{\in}\right)^n+\frac{\omega_{n-1}n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}}\int_{\in}^{1} r^{n-1}\log^n r\,dx\right)^{\frac{n}{n-1}}}.$$

We now ask that L satisfies

$$\frac{c^{\frac{n}{n-1}}}{L^n}\to 0,\text{ as } \in\to 0. \qquad (24)$$

Then, for $\in>0$ sufficiently small, we have

$$-\frac{\propto_n^{n-1}}{(n-1)!}\frac{1}{1+\frac{\omega_{n-1}}{n}c^n\left(L_{\in}\right)+\frac{\omega_{n-1}n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}}\int_{\in}^{1} r^{n-1}\log^n r\,dx}$$

$$+\frac{\propto^n}{n!}\frac{\omega_{n-1}L^n / c^{\frac{n^2}{(n-1)^2}}\int_{\in}^{1} r^{n-1}\log^{\frac{n^2}{n-1}} r}{\left(1+\frac{\omega_{n-1}}{n}c^n\left(L_{\in}\right)^n+\frac{\omega_{n-1}n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}}\int_{\in}^{1} r^{n-1}\log^n r\,dx\right)^{\frac{n}{n-1}}}$$

$$\geq B_1 L^{n-\frac{n^2}{n-1}}-B_2\frac{c^{\frac{n}{n-1}}}{L^n}$$

$$=\left(\frac{c^{\frac{n}{n-1}}}{L^n}B_1\frac{L^{2n-\frac{n^2}{n-1}}}{c^{\frac{n}{n-1}}}-B_2\right),$$

where B_1, B_2 are positive constants.

When $n>2$, we may choose $L=bc^{\frac{1}{n-2}}$; then, for b sufficiently large, we have

$$B_1\frac{L^{\frac{n}{n-1}(n-2)}}{c^{\frac{n}{n-1}}}-B_2=B_1 b^{\frac{n}{n-1}(n-1)}-B_2>0,$$

And (24) holds. Thus, we have proved that for $\in>0$ sufficiently small

$$\int_{R^n}\Phi\left(\propto_n\left(\frac{u_{\in}}{\|u_{\in}\|_{H^{1,n}}}\right)^{\frac{n}{n-1}}\right)dx>\frac{\propto_n^{n-1}}{(n-1)!}.$$

Corollary (2.3) (Shawgy and Mahgoub, 2011): Prove that for $\in>0$

$$\int_{R^n}\Phi\left(u_{\in}\right)^{n/n-1}dx>\frac{1}{(n-1)!}.$$

Proof: For $k = \in > 0$ Results (5.1.8)(ii) implies that $\|u_\in\| < \frac{\alpha}{\beta^{n/n-1}}$. If $u_\in \to u \in H^{1,n}(R^n)$ *with* $\|u\|_{H^{1,n}(R^n)} = 1$, we have for $\alpha_n \to \alpha$ that $\beta < \alpha^{n/n-1}$. Therefore

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_\in}{\|u_\in\|_{H^{1,n}}} \right)^{\frac{n}{n-1}} \right) dx = \int_{R^n} \Phi(\alpha u \in)^{n/n-1} d\,x > \frac{\alpha_n^{n-1}}{(n-1)!} = \frac{\alpha^{n-1}}{(n-1)!} > \frac{\beta}{(n-1)!}.$$

Hence $\int_{R^n} \Phi(u_\in)^{n/n-1} dx > \frac{1}{(n-1)!}$.

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