Nile Valley University Publications
Nile Journal for Sciences and Engineering (NJSE)
(ISSN: 1858 – 7059)
Volume01, NO. 02, 2024
https://www.nilevalley.edu.sd

On The Topological Groups and Their Compactifications

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Abstract

In this paper some topics in topological groupshas beendiscussed, and the compact spaces and compactifications of topological groups were stated.Firstly, many definitions have been statedand followed by many examples of topological groups, some theorems have been included which are propositions and lemmas as well asthe locally compact abelian topological groupswere discussed.Secondly, the paper included a compactification of topological groups and prove some theorem and some propositions concern this topic.

Keywords:Topological Groups, Compact Spaces, Abelian Topological Groups, Locally Compact, Compactifications, Group Homeomorphism.

حول الزمر الطوبولوجية وتراصها

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مُسْتَخْلَص

تم في هذه الورقة مناقشة بعض الموضوعات الخاصةبالزمر الطوبولوجية، كما شملت الفضاءات المتراصة والتراصيات في الزمر الطوبولوجية. أولاً تم إيراد العديد من التعريفات، وأُتبعت بالكثير من الأمثلة للزمر الطوبولوجية. شملت الورقة أيضاً بعض النظريات المتمثلة في المبرهنات والتمهيديات علاوة على مناقشة الزمر الطوبولوجية الابلية المتراصة محليا. ثانياً حوت الورقة التراصيات بالنسبة للزمر الطوبولوجية مع تقديم براهين لبعض النظريات والتمهيديات التي تخص هذا الموضوع

كلمات مفتاحية: *الزمر الطبولوجية، الفضاءات المتراصة، الزمر الطوبولوجية الابلية، التراص المحلي، التراصيات، الزمر التماثلية.*

Introduction

The topology science is very important science, it has many efficient in mathematical field, and it is entering in many sciences, like physics and engineering. The important topics in topology science, are topological spaces, topological groups and compactness, which play a very important part in pure mathematics. Every group, can be made into a topological group, imposing the discrete topology on it. The topological groups have many parts, like abelian topological groups and non-abelian topological groups and topological transformation groups. The compactness and compactifications are very important in topology science, because they are referring to topological spaces and topological groups, they can be described as the maximal ideal spaces of certain functions algebras.

1. Topological groups

In this paper, we discuss some definitions and some examples of topological groups, and we state some theorems, lemmas and some propositions on topological groups. In the last section, someexplanations about locally compact abelian topological groups, with refer to [1],[3] and [5].

2.1 Definitions

The definitions below are included in [1] and [3].

Definition (2.1.1): A topological group is a triple (G,τ), where (G,) is a group and τ is a topology on G such that, the function $f: G \times G \to G$defined by$f(x, y) = x . y^{-1}$ for $x, y \in G$ is continuous.

Here $G \times G$ is viewed as a topological space by using the product topology. It is common to require that the topology on G be Hausdorff.

Definition (2.1.2): We say that (G, X,τ) is a topological group if (G, X) is a group and (G,τ) is a topological space such that, writing $M(x, y) = x \times y$ and $Jx = x^{-1}$ the multiplication map $m : G^2 \to G$ and the inversion map $J: G \to G$ are continuous.

Definition (2.1.3): Let G be a topological group, and let $a \in G$, then:

i. The map $L(a): G \to G : x \to ax$ is called a left translation.
ii. The map $R(a): G \to G : x \to xa$ is called a right translation.

Proposition (2.1.4): Let G be a topological space, which is also a group, then G is a topological group if and only if:

i. The set $\{e\}$ is closed.
ii. For all $a \in G$ the translations $R(a)$ and $L(a)$ are continuous.
iii. The mapping$G \times G \to G : (x, y) \to xy^{-1}$ is continuous at the point(e, e).

Definition (2.1.5): If (G,X_G ,τ_G) and (G,X_H ,τ_H) are topological groups, we say that, $\theta: G \to H$is an isomorphism if it is a group isomorphism and a topological homeomorphism.

Lemma (2.1.6): let Ube neighborhood of$e$ in a topological group G, then there exists a neighborhood$v$ of e such that $v \subset U$ and $v = v^{-1}$ and $vv = vv^{-1} \subset U$. We shall call such a neighborhood v of e symmetric.

Definition (2.1.7): A local group is a Hausdorff space N such that:

i. There is a binary operation in N , $(x, y) \to xy$ which is defined for certain pairs $(x, y) \in N \times N$.
ii. The operation is associative.
iii. There exists $e \in N$, thus for all $x \in N$, $xe = ex = x$.
iv. There exists an inverse operation in N , $X \to X^{-1} : xx^{-1} = x^{-1}x = e$.
v. The maps $(x, y) \to xy$ and $x \to x^{-1}$ are continuous.

Definition (2.1.8):

i. The local groups N and N' are topologically isomorphic if there exists a homeomorphism $f: N \to N' : x \to f(x)$, such that the product xy is defined in N if and only if the product $f(x)f(y)$ is defined in N', andin this case $f(xy) = f(x)f(y)$.
ii. The topological groups G and G' are locally isomorphic, they have open nuclei which is local group, are topologically isomorphic.

Definition (2.1.9): Topological space M is a $T_0 - space$, if for any given pair of distinct points $x, y \in M$, there exists an open set U of M, which contains one of the points, but not the other.

Definition (2.1.10): A $T_0 -$ topological group is a group G which is $T_0 - space,$ and such that, the map, $G \times G \to G : (x, y) \to xy^{-1}$, is continuous.

Definition (2.1.11): A topological space M is a $T_1 - space$ if for distinct points $x \neq y\ in\ M$, there exists an open set $V\ with\ y \in V\ but\ x \in V$.

Definition (2.1.12): Let M a Hausdorff topological space, and let G be a topological group, then:

i. G operates on M if there is, a surjection $G \times M \to M : (g, p) \to g.p$ Such that $(g_1 g_2).p = g_1.(g_2.p)$, and $e \cdot p = p$ for all$g_1, g_2 \in G$ and $p \in M$ where e is identity of G.
ii. G operates transitively on M , if for every $p, q \in M,$ there exists $g \in G$Such that $g \cdot p = q$.
iii. G operates continuously on M , if the map $G \times M \to M : (g, p) \to g.p$ is continuous.
iv. G is called a topological transformation group on M , if G operates Continuously on M.
v. G is effective if $a \cdot p = p,$ for all $p \in M$ emplies $a = e$.
vi. Let p be fixed in M , then $G(p) = \{g \in G : g \cdot p = p\}$ is a group called the isotropy subgroup of G at p. The set $G \cdot p = \{g \in G : g \cdot p = p\},$ is called an orbit under G.

Definition (2.1.13): A homogenous space M is a space with a transitive group action by Lie group action implies that there is only one group orbit, M is isomorphic to quotient space G/H, where H is the isotropy group G.

2.2 Examples

Example (2.2.1): Examples of abelian topological groups

Before we state the examples, we assert that; every group can made into a topological imposing the discrete topology on it. Here are some important examples of an abelian topological groups:

1. All Euclidean spaces under the usual additional, are abelian topological groups.

2. The non-zero real numbers, or the non-zero complex numbers under multipli-cation form an abelian topological group.
3. All topological vector spaces, such as Banach spaces, or Hilbert spaces are abelian topological groups.
4. Let X be a space and $H(X)$, is the group of all homeomorphisms of X . If $X$is locally compact and regular, then $H(X)$, becomes an abelian topological group under composition.
5. If $\{G_i : i \in I\}$ is a collection of topological groups, the product space $\prod_{i \in I} G_i$ can be made into topological group under co-ordinate wise multiplication.
6. Let H a subgroup of a topological group G and $G/H$is the space of right cosets of$H$ in G then, if H is a normal subgroup of G , then G/H is a topological group.

Examples (2.2.2): of non-abelian topological groups

All examples and properties below are in (Arther and Ralph, 1973).

Consider the subgroup of rotations of R^3, generated by two rotations by irrational of multiplies of 2π about different axes.

All the above examples are Lie groups (topological groups that are also manifolds).

An example of a topological group which is not Lie group is given by the rational numbers Q ,this a countable space, and it does not have the discrete topology.

2.3 Properties:

1 If a is an element of a topological group G,then left or right multiplication with $a$yields a homeomorphism$G \to G$, thus can be usedto show that all topological groups are actually uniform spaces. Every topological group can be viewed as a uniform space in two ways, the left uniformly turns all left multiplication into uniformly continuous maps, while the right uniformly turns all right multiplication into uniformly continuous maps. If G is not abelian, these two need not coincide.

2 As a uniform space, every topological group is completely regular. It follows that if a topological group is $T_0(i.e. Kolmogorov)$,then it is already T_2 (i.e. *Hausdorff*).

3 The most natural notion of homeomorphism between topological groups is that of a continuous group homeomorphism. Topological groups, together with continuous group homeomorphisms as morphemes, form a category.

4 If H is a normal subgroup of the topological group G, then, the factor group G/H becomes a topological group by using the quotient topology (the finest topology onG/Hwhich makes the natural projection $G \rightarrow G/H$ continuous).

5 The algebraic and topological structures of a topological group interact in non-trivial ways, for example, in any topological group the connected component containing the identity element, is a normal subgroup.

2.4 Subgroups and nuclei

Definition (2.4.1): Let G be a topological group, and let H be a subset of G such that, $HH^{-1} \subset H$, then H is a subgroup of G .(Arther and Ralph, 1973)

Corollary (2.4.2): Let H be a closed normal subgroup of the topological group$G$ and G/H is a quotient, then G/H becomes a topological group such that the projection $\pi: G \rightarrow G/H$ is an open continuous homeomorphism. (Arther and Ralph, 1973)

Definition (2.4.3) (Arther and Ralph, 1973**):** Let G be a topological group, then the center$C$ of G equals $\{x \in G : xa = ax \ \forall \, a \in G\}$. The center is a normal subgroup of G,and is also denoted by $Z(G)$. (Arther and Ralph, 1973)

Definition (2.4.4): Discrete subgroups of Euclidean spaces under usual additional are known as Lattices. (see [1])

Definition (2.4.5): Let (G, X, T) be a topological group, and H a subgroup of *G*: (Torner, 2003)

i. The topological closure $\overline{H}$ of H is a subgroup.
ii. If H is normal, so is $\overline{H}$.
iii. If H contains an open set, then H is open.
iv. If H is open, then $\overline{H}$ is closed.
v. If H is closed and of finite index in G , then $\overline{H}$is open.

Lemma (2.4.6) : Let (G, X, T) be a Hausdorff topological group, if H is a closed normal subgroup, then G/H is Hausdorff. (see [1])

Lemma (2.4.7): Let (G, X, T) be a topological group, then $I = \{\overline{e}\}$ is a closed normal subgroup. (see [1])

Definition (2.4.8) : If G is a topological group, the connected component of identity $e \in G$ is called the identity component of G, and is denoted by G_0 , and $G_0$is closed normal subgroup of G , and the connected component $c(a)$ of $a \in G$ equals aG_0. (Arther and Ralph, 1973)

Definition (2.4.9): A subset of atopological group$G$,which contains an open neighborhood of the identity e, is called a nucleus of G. (Arther and Ralph, 1973)

Proposition (2.4.10): let V be the family of all nucleus of a topological group G, then V satisfies: (see [1])

i. $v_1 , v_2 \in$ V implies $v_1 \cap v_1 \in$ V ;
ii. $v_1 \in$ V^2 and $v_1 \subset W \subset G$ implies $W \in$ V;
iii. For any $v_1 \in$ V , there exists $v^{-1} \in$ V such that $vv^{-1} \in v_1$;
iv. If $v \in$ V and $a \in G$, then $v^{-1} \in$ V ;
v. $v^{-1} \cap \{v: v \in$ V$\} = \{e\}$.

2.5 Locally compact abelian topological groups

Let A be an abelian group, thus A is a set equipped with a binary operation + , which is commutative and associative, there is identity element $0 \in A$, such that $0 + a = a$ for all $a \in A$, and each $a \in A$ has an inverse $-a$, characterized by $a + -a = 0$, as a basic examples, the integers, real numbers and complex numbers are abelian groups under addition, and for each positive number n we have the integers modulo n , a cyclic group with n elements.

Let us also assume that A is a topological space, which is to say that certain subsets of A are designated as open subsets. As usual one required that the empty set and A itself are open subsets of A , that the intersection of any finite collection of open subsets is an open subset, and that the union of any collection of open sets, is an open subset. Once the open subsets are selected, the closed subsets are defined to be the complements in A of the open subsets.

Various standard notions, such as continuity at a point of a mapping between two topological spaces, can be defined in terms of the open subsets through standard methods.

To say that A is a topological group, means that the group structure and topo-logy are compactible in a natural way. Specifically, the group operation + should be continuous as a mapping from $A \times A$ into A, and $a \to -a$, should be continuous as a mapping from A to it self. This implicitly uses the product topology on$A \times A$ is open if it is the union of products of open subsets of A. It is customary to require that A be a Hausdorff topological space, which is equivalent in this setting to the requirement that $\{0\}$ be a closed subset of A.

In any topological space a subset K is said to be compact if every open covering of K in the space admits a finite sub covering, i.e., if for every family$\{U_i\}_{i\in I}$ of open subsets of the topological space such that:

$(i) K \subseteq U_{i\in I} u_i.$

There is a finite collection $i_1, \cdots, i_I$of indices in I such that:

$(i) K \subseteq U_{iI} u, \cdots u U_{iI}.$

A topological space is said to be locally compact if for each point x in the space, there is an open subset W and a compact subset K of the space such that:

$$(i) x \in W \subseteq K.$$

A locally compact abelian topological group is an abelian topological group which is locally compact as a topological space.

Of course, local compactness at the identity element e implies local compact-ness at every point because group translations define homeomorphism.

As in (3), it is simpler to say "LCA group" in place of locally compact abelian topological group. The integers modulo$n$ are natural examples of LCA groupequipped with their discrete topologies, in which every subset is considered to be open. For the real and complex numbers, one can use their standard topologies, indeed by the usual Euclidean metrics, to get LCA group. One can also consider the nonzero complex numbers using multiplication as the group operation and the usual topology. If one takes the complex numbers with modulus 1 using multiplication as thegroup operation and the usual topology, one gets acompact LCA group.

Fix an integer $n \geq 2$, and consider the group consisting of sequences $x = \{x_j\}_{j-1}^{\infty}$ such that each of them is an integer such that $0 \leq x_j \leq n-1$, and the sum of two elements x, y in the group A is defined by adding each term modulo n. If x is an element of the group and$i$ is a positive integer, then the ith standard neighborhood around$x$ is defined to be the set of y in the group such that $x_j = y_j$, then $1 < j < I$. This leads to a topology on the space in which a sub set of the group is open, if for each point x which is contained in the subset. Well known results in topology imply that this space is compact with respect to this topology, and in fact it is homeomorphic to the cantor set.

It is easy to see that this example defines a compact LCA group. Namely, the group operations are continuous with respect to the topology just defined. This is a nice example where the topological dimension is equal to e ,which isto say that the space is totally disconnected, with no connected subsets with at least two elements. As the same time the topology is not the discrete topology.

Let A be a LCA group. A basic object of interest associated to A is translation-invariant integral, which is a linear mapping from the vector space of complex-valued continuous functions as $f(x)$ on A with compact support into the complex number such that the integral of $f(x + a)$ is equal to the integral of $f(x)$ for all a ∈ A , the integral of a real valued function is a real number, the integral of a non-negative real number, and the integral of f is positive, if f is a non-negative real valued continuous function on A with compact support such that $f(x) > 0$ for some $x \in A$. In the examples described earlier such that an integral can be defined explicitly, in terms of sums, classical Riemann integrals, or simple generalizations of Riemann integrals for the spaces of sequences modulo n . A general theorem states that any LCA group$A$ an invariant integral, and that this integral is unique except for multiplying it by a positive real number.

Let A be an abelian group. By a character on A we mean a continuous group homeomorphism from A into the group of complex numbers with modulus I with respect to multiplication. Sometimes one may wish to consider unbounded characters more generally, which are continuous homeomorphisms from$A$ into the non-zero complex numbers with respect to multiplication. Note that any bounded subgroup of the nonzero complex numbers with respect to multiplication is contained in the complex numbers with modulus I as one can easily verify. Thus, a bounded, continuous homeomorphism from A into the group of nonzero complex numbers is a character, and in particular every continuous homeomorphism from A into nonzero complex numbers is a character when A is compact.

If $f(x)$is a complex valued continuous function on A with compact support, or an integrable function more generally, one can define its Fourier transformation $\hat{f}(\emptyset)$by saying that if $\emptyset$ is a character on A , then $\hat{f}(\emptyset)$is the integral of f times the complex numbers conjugate of $\emptyset$, using a fixed invariant integral on A as discussed previously. If A is not compact, then one can extend this to a Fourier – Laplace transform by allowing unbounded characters, at least when f has compact support or sufficient integrality properties. For bounded character one has the usual inequality which states that $\left|\hat{f}(\emptyset)\right|$ is less than or equal to the integral of $|f|$.

Many classical a sects of Fourier analysis work in this setting. A basis point is that the Fourier transform diagonalizes translation operator, which means that if $a \in A$ and $f(x)$ is continuous function on the group with compact support, or an integrable function on the group, then the Fourier transform of $f(x-a)$at the character $\emptyset$ is equal to $\overline{\emptyset(a)}$ times as Fourier transform of $f(x)$at $\emptyset$. One can also define convolution in the usual way, $\oplus$using the invariant integral on A , and the Fourier transform of a convolution is equal to the product of the corresponding Fourier transforms.

3 Topologicalgroups' compactificationsIntroduction

Every topological group G has some natural compactifications. They can bedescribed as the maximal ideal spaces of certain functions algebras, or as the Samuel compactifications for certain uniformities on G , some compactifications of G carry an algebraic structure, and may be useful for studying the group G itself.

We consider, in particular, the following constructions: the greatest ambit $S(G)$ and the universal minimal compact$G$-space M_G(sections 2 and 3); e Roelcke compactifications $R(G)$ (section 4) ; the weakly almost periodic compactifications $W(G)$(section 5). In the last case the canonical map $G \to W(G)$ need not be an embedding. In section 6 , we discuss the group of isometrics of the Urysohn universal metric space u, all these topics are included in [4] and [5].

3.2 Greatest ambit $S(G)$

Let G be a topological group, the Banach space $B = RUC^b(G)$ of all right uniformly continuous bounded complex functions on G is a $C^* - algebra$, and G acts on B by $C^* - algebra$ automorphisms. Let $S(G)$be the compact maximal ideal space of B . It is the least compactifications of G over which all functions from B can extended. The topological group of all $C^* - algebra$

automorphisms of$B$ is naturally isomorphic to$H\big(S(G)\big)$. It follows that G acts on $S(G)$, and the natural homeomorphism$G \to H(S(G))$ is a topological embedding.

The space $S(G)$ can also be described as the Samuel compactifications of the uniform space (G, R) , here R is the right uniformity on G . The basic entourages for R are of the form $\{(x, y) \in G : xy^{-1} \in V\}$, where $V \in N(G)$. The Samuel compactifications of a uniform space (x, u) is the completion of x with respect to the finest pre-compact uniformity which coarser than u .

We shall consider G as dense subspace of $S(G)$. The action $G \times S(G) \to S(G)$ extends the multiplication $G \times G \to G$.

A $G - space$ is a topological space X with a continuous action of G , that is, a map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $ix = x$, $(g, h \in G, x \in X)$. A $G - map$ is a map $f : X \to Y$ between $G - spaces$ such that$f(gx) = gf(x)$ for all $x \in X$, $g \in G$. The $G - spacesS(G)$ has distinguished point e (the unity), and the pair $(S(G), e)$ has the following universal property : for every compact $G - space$ X and every $p \in X$ there exists a unique $G -$ map $f: S(G) \to X$ such that $f(e) = p$. Indeed, the map$g \to gp$ from G to X is $R -$ uniformly continuous and hence can be extended over $S(G)$.

Theorem (3.2.1) : For every topological group G the greatest ambit $X = S(G)$ has a natural structure of a left topological semi group with a unity such that the multiplication $X \times X \to X$ extends the action $G \times X \to X$.

Proof

Let $x, y \in X$ such that $r_y(e) = y$. Define $xy = r_y(x)$. Let us verify that the multiplication $(x, y) \to xy$ has the required properties. For a fixed y the map $x \to xy$ is equal to ry and hence is continuous. If $yz \in X$, the self-maps$r_z r_y$ and r_{zy} of X are equal, since both are $G -$ maps sending e to $yz = r_z(y)$. This means the multiplication on nX is associative. The distinguished element $e \in X$ is the unity of X : we have $= r_x(x) = x$. If $g \in G$ and $x \in X$, the expression gx can be understood in two ways: in the sense of exterior action of G on X as a product in X.To see that these two meanings agree, note that $r_x(g) = r_x(ge) = gr_x(e) = gx$ (the exterior action is meant in the last two terms; the equality holds since r_x is a $G - map$.

3.3 Universal minimal compact $G - space$

Definition (3.3.1):A $G - space$ is minimal if it has no proper$G -$ invariant closed subset or, equivalently, if the orbit Gx is dense in X for every $x \in X$.

Lemma (3.3.2) : The Universal minimal compact $G - space$ M_G is characterized by the following property: M_G is minimal compact $G - space$ and for every compact minimal $G - space\ X$ there exists a $G - map$ of M_G onto X.

Since $\text{Zorn}'\text{s}$ lemma implies that every compact $G - space$ has a minimal compact $G - subspace$, it follows that for every compact $G - space\ X,$ minimal or not, there exists a $G - map$ of M_G to X.

Lemma (3.3.3): The existence **of** M_G is easy: take for M_G any minimal closed $G - subspace$ of $S(G)$ universal property of $(S(G), e)$ implies the correspond-ding universal property of M_G . It is also true that M_G is unique, in the sense that any two Universal minimal compacts$G - spaces$ are isomorphic.

Proposition (3.3.4): If $f: X \to X$ is a $G - self - map$and $a = f(e)$then $f = r_a$.

Proof: We have $f(x) = f(xe) = x\big(f(e)\big) = xa = r_a(x)$ and hence for all $x \in X$.

A subset $I \subset X$ is a left ideal if $XI \subset I.$ Closed $G - \text{subspaces}$ of X are the same as closed left ideals of$X$. An element$x$ of asemi group is an idempotent if $x^2 = x$. Every closed $G -$subspace of X , being aleft ideal, is moreover a left topological compact and hence contains an idempotent.

Theorem (3.3.5): Every non-empty compact left topological semi group K contains an idempotent.

Proof: Zorn′s lemma implies that there exists a minimal element Y in the set of all closed non-empty sub semi groups of K. Fix $a \in Y$. We claim that $a^2 = a$(and hence Y is a sigleton). The set Ya, being aclosed semi group of Y. It follows that the closed sub semi group $Z = \{x \in Y : xa = a\}$ is non-empty. Hence $Z = Y$ and $xa = a$ for every $x \in Y$. It particular, $a^2 = a$.

Let M be a minimal closed left ideal of X. We have just proved that there an idempotent $P \in M$. Since XP is closed left ideal contained in M, we have $XP = M$. It follows that $xp = x$ for every $x \in M$. The $G - map$ $r_p : X \to M$defined by$R_P(X) = XP$ is a retraction of x onto M.

Proposition (3.3.6): Every $G - mapf : M \to M$ has the form$f(x) = xy$ for some $y \in M$.

Proof: The composition $h = fr_P: X \to M$ is a $G - map$ of X into itself, hence it has the form $h = r_y$,where $y = h(e) \in M$ (Proposition 3.3.1). Since $r_P \upharpoonright M = Id,$ we have $f = h \upharpoonright M = r_y \upharpoonright M$.

Proposition (3.3.7): Every $G - mapf : M \to M$ is bijective.

Proof: According to Proposition (3.3.6), there is $a \in M$ such that $f(x) = xa$ for all $x \in M$. Since Ma is a closed left ideal of X contained in M, we have $Ma = M$ by the minimality of M. Thus there exists $b \in M$ such that $ba = p$.Let every$g : M \to M$ be the $G -$ map defined by $g(x) = xb$. Then $fg(x) = xba = xb = x$ for every $x \in M$, and therefore $fg = I$ (the identity map of M). We have proved that in the semi group S of all $G - self - map$ of M, every element has a right inverse. Hence S is a group (alternatively, we first deduce from the equality f g=*1* that all elements of S are surjective and then, applying this to g , we see that f is also injective.)

Theorem (3.3.8): For every topological group G the action of G on the universal minimal compact $G - space$ M_G is not 3-transitive.

For example, if K is a compact manifold of dimension> 1, or a compact Menger manifold and $G = H(K)$,then $M(G) \neq K$, since the action of G on K is 3-transitive.

It would be interesting to understand what is $M(G)$ in this case.

Let P be the pseudo arc (= the unique hereditarily indecomposable chainable continuum) and $G = H(P)$. The action of G on P is transitive but not 2-transitive, and the following question remain open:

Let P the pseudo arc and $G = H(P)$. Can M_G be identified with P ?

Question (3.3.9): Let G be abelian topological group. Suppose that G has no non-trivial continuous characters $X : G \to T$. Is G extremely amenable.

For cyclic group the question can be reformulated as follows: Let K be a compact space, and let $f \in H(K)$ be a fixed-point free homeomorphism of K . Let G be the cyclic subgroup of $H(K)$ generated by f . Does there exists a complex number a such that $|a| = I,\ a \neq I$, and the homeomorphism $X : G \to T$ defined by $X(f^n) = a^n$ is continuous.

If K is a circle, the answer is yes: for every orientation-preserving homeomorphism$f$ of a circle, the rotation number is defined which gives rise to a non-trivial continuous character on the group generated by f .

A positive answer to question (3.3.9) would imply the solution of the problem: Is it true that for every big set S of integers, the set $S - S$ contains a neighborhood of zero for Bohr topology on Z ? A set S of integers is said to be big (or syndetic) if $S + F = Z$ for some finite $F \subset Z$; this means that the gaps between consecutive terms of$S$ are uniformly bounded. The Bohr topology on$Z$ generated by all characters $X : Z \to T$. It is known that for every big subset $S \subset Z$ the $\mathrm{S} - \mathrm{S} + \mathrm{S}$ contains a Bohr neighborhood of zero.

Extremely amenable groups can be characterized in terms of big sets. A subset S of a topological group G is big on the left, or left syndetic, if $FS = G$ for some finite $F \subset G$.

Theorem (3.3.10): A topological group$G$is extremely amenable if and only if whenever $S \subset G$ is big on the left, SS^{-1} is dense in G.

Theorem (3.3.11): A topological group G is extremely amenable if and only if for every bounded left uniformly continuous function f from G to a finite dimensional Euclidean space, every $\varepsilon > 0$, and every finite (orcompact)$K \subset G$ there exists $g \in G$ such that diameter $f(gk) < \varepsilon$.

3.4 Roelcke Compactifications

Definition (3.4.1): For a topological group$G$let $R(G)$ be the maximal ideal of the $C^* -$ algebra of all bounded complex functions on G which are both left and right uniformly continuous. The space $R(G)$ is the Samuel compactifications of theuniform space$(G, \ell \wedge R)$, where ℓ is the left uniformly on G, R is the rightuniformly, and $\ell \wedge R$ is the Roelcke uniformly on G , the greatest lower bound of ℓ and R . We call $R(G)$ Roelcke compactifications of G .

Lemma (3.4.2): While the greatest lower bound of two compactible uniformities on a topological space in general need not be compatible, the Roelcke uniformity is compatible with the topology of G . The covers of the form $\{UxU : x \in G, U \in N(G)\}$, constitute a base of uniform covers of Roelcke uniformity.

If G is abelian, $(G) = S(G)$. In general,$R(G)$ is a $-space$, and the identity map of G extends to a $G - map\ S(G) \to R(G)$.

Definition (3.4.3): The group G is precompact if one of the following equivalent properties holds:

i. (G, L) is precompact.
ii. (G, R) is precompact.
iii. G is a subgroup of a compact group.

It can be shown that G is precompact if and only if G for everyneighborhood U of unity, there exists a finite $F \subset G$ such that $G = F \cup F$. Let us say that G is Roelcke precompact if the Roelcke uniformity $\ell \wedge R$ is precompact. This exists a finite $F \subset G$ such that $G = F \cup F$. There are many non-abelian non-precompact groups which are Roelcke compactifications. For example, the symmetric group $symm\,(E)$ of all permutations of a discrete space E , or the unitary group $U(H)$, on a Hilbert space H , equipped with the strong operator topology, are Roelcke precompact. The Roelcke compactifications of these groups can be explicitly described with the aid of the following construction:

Suppose that G acts on a compact space K . For $g \in G$, let $r(g) \subset K^2$ be the graph of the $g -$ shift $x \to gx$. The map $g \to \Gamma(g)$ from G to $Exp\, K^2$ is both left and right uniform continuous (if the compact space $Exp\, K^2$ is equipped with its unique compactible uniformity), hence it extends to a map $f_k : R(G) \to Exp\, K^2$. If the action of G on K is topologically faithful, the map f_k often happens to be an embedding in which case $R(G)$ can be identified with the closure of the set $\{\Gamma(g): g \in G\}$ in $Exp\ K^2$. For example, this the case if $K = S(G)$ or $K = R(G)$.

The space $Exp\ K^2$ is the space of all closed relations on K. It has a rich structure, since relations can be composed, reversed, or compared by induction. This structure is party inherited by $R(G)$. Let us consider some examples.

Example (3.4.4): Let $G = symm(E)$ be the topological symmetric group**.** It acts on the compact cube$K = 2^E$ **.** The natural map$f_k : R(G) \to Exp\ K^2$in an embedding.

Example (3.4.5): Let Gbe the unitary group$U(H)$, of a Hilbert space$H$, equipped with the strong operator topology (this is the topology of point wise convergence inherited from the product H^H). Let K be the unit ball of H . Equip K with the weak topology. Then K is compact. The unitary group G acts on K , and the map $R(G) \to Exp\ K^2$ is an embedding.

The space $R(G)$ has a better description in this case : $R(G)$ can be identified with the unit ball θ in Banach algebra $B(H)$ of all bounded linear operators on H. The topology on θ is the weak operator topology: the map $A \to A/K$ which assigns to every operator of norm ≤ 1 its restriction to K is a homeomorphic embedding of θ into the compact space K^K. Thus $R(G)$ has a natural structure of semi topological semi group.

Example (3.4.6): Let K be a zero-dimensional compact space such that all non-empty clopen subsets of K are homeomorphic to K . Let $G = H(K)$, the natural map $f_k : R(G) \to Exp\ K^2$ is an embedding. Moreover, the image of f_k , which is the closure of the set of all graphs of self-homeomorphisms of K , is the set θ of all closed relations on K whose domain and range are equal to K . Thus $R(G)$ can be identified with θ .

This time $R(G)$ is an ordered semi group, but not a semi topological semi group, since the composition of relations is not a separately continuous operation. As in the pervious example, one can use the space $R(G)$ to prove that G is minimal. Moreover, every non-constant onto group homeomorphism $f : G \to H$ is an isomorphism of topological groups. To prove this, we proceed as before extend f to $f : G \to H$ and look at the kernel $S = F^{-1}(e_n)$. Zorn's lemma implies the existence of maximal idempotent in S(with respect to the inclusion). Symmetric idempotent above the unity (= the identity relation = the diagonal of K^2) in $\theta$are precisely closed equivalence relations on K . Since there are no non-trivial choices for S either $S = \{1\}$ or $S = \theta$.

Example (3.4.7): Let $G = H_+(1)$be the group of all orientation-preserving homeomorphisms of the closed interval $I = [0,1]$. The map $f_G : R(G) \to Exp\ I^2$ is a homeomorphic embedding. Thus $R(G)$ can be identified with the closure of the set of all graphs of strictly increasing functions $h : I \to I$ such $h(0) = 0$ and $h(1) = 1$.

This closure consists of all curves $C \subset I^2$ which lead from (0,0) to (1,1) and like graphs of increasing functions, with the exception that$C$ may include both horizontal and vertical segments.

There seems to be no natural semi group structure on $R(G)$. This observation leads to an important result: The group G has no non-trivial homeomorphisms to compact semi topological semi groups and has no non-trivial representation by isometries in reflective Banach space.

3.5 WAP compactifications

Definitions (3.5.1): Let S a semi group and a topological space. if the multiplication $(x, y) \to xy$ is separately continuous (this means that the maps $x \to ax$ and $x \to xa$ are continuous for every $a \in S$), we say that S is a semi topological semi group.

For a topological group G let $f : G \to W(G)$ be the universal object in the category of continuous semi group homeomorphisms of G to compact semi-topological semi groups. In other words, $W(G)$ is a compact semi topological semi group, and for every continuous homeomorphism $g : G \to S$ to a compact semi topological semi group$S$ there exists a unique homeomorphism $h: W(G) \to S$ such that $g = hf$.

The existence of$W(G)$follows from twofacts: (1) arbitrary products are defined in the category of compact semi topological semi groups; (2) the cardinality of a compact space has an upper bound in terms of its density. The space $W(G)$can also be defined in terms of weakly almost periodic functions. Recall the definition of such functions.

Let a topological group G act on a space X . Denote by $C^b(X)$ the Banach space of all bounded complex valued continuous functions on Xequipped with the supremum norm. A function $f \in$

$C^b(X)$ is called weakly almost periodic ($w.a.p.$ for short), if the G-orbit of f is weakly relatively compact in the Banach space $C^b(X)$.

In particular, considering the left and right actions of a group G on it self, we can define left and right weakly almost periodic functions on G . These two notions are actually equivalent, so we can simply speak about $w.a.p.$ functions on a group G. The space WAP of all $w.a.p.$ functions on a group G is a $C^* -$ algebra , and the maximal ideal space of this algebra can be identified with $W(G)$.Thus the algebra WAP is isomorphic to the algebra $C\big(W(G)\big)$of continuous functions on $W(G)$. We call $W(G)$ the weakly almost periodic $w.a.p.$ compacti-fications of the topological group G .

Remark (3.5.2): We show a compactification of a topological space X, we have a compact Hausdorff space Ktogether with a continuous map$j : X \to K$

with a dense range. We do not require that j be a homeomorphic embedding.

For every reflexive Banach space X , there a compact semi topological semi group $\theta(X)$ associated with X : the semi group of all linear operators $A : X \to X$ of norm ≤ 1 , equipped with the weak operator topology. Recall that a Banach space X is reflexive if and only if the unit ball B in X is weakly compact. If X is reflexive, $\theta(X)$ is homeomorphic to a closed sub space of B^B(where$B$ carries the weak topology, and hence compact) .

It turns out that every compact semi topological semi group embeds into $\theta(X)$ for some reflexive X .

Theorem (3.5.3): Every compact semi-topological semi-group is isomorphic to a closed sub semi group of , $\theta(X)$ for some reflexive Banach space.

The group of invertible elements of $\theta(X)$ is the group $Is_w(X)$of isometries of X, equipped with the weak operator topology. This topology actually coincides with the strong operator topology.

Theorem (3.5.4): For every reflexive Banach space, the weak and strong operator topologies on the group $Is(X)$ agree.

In particular, the group of invertible elements of $\theta(X)$ is a topological group. The natural action of this group on $\theta(X)$is (jointly) continuous. This can be easily deduced from the fact that the topological group $Is_S(X^*) = Is_w(X^*)$ are canonically isomorphic. In virtue of Theorem (3.5.3), similar assertions hold true for every compact semi topological semi group S : the group$G$ of invertible elements of$S$is atopological group, and the map$(x, y) \to xy$ is joint continuous on $G \times S$. Thus S is a G-space.

It follows for every topological group G the compact semi topological semi group $W(G)$ is a G-space , hence there exists a G-map $S(G) \to W(G)$ extending the canonical map $G \to W(G)$In terms of function algebras, this means that every $w.a.p.$ function on G is right uniformly continuous , since the algebra WAP is invariant under the inversion on G , $w.a.p.$ functions are also left uniformly continuous and hence Roelcke uniformly continuous .

It follows that there is a natural map $R(G) \to W(G)$. If $G = U(H)$ is the unitary group of a Hilbert space H , then $R(G) = \theta(H)$ is a compact semi topological semigroup,and therefore the canonical map $R(G) \to W(G)$ is a homeomorphism, thus $W(G) = \theta(H)$ .The canonical map$S(G) \to W(G)$ is a homeomorphism if and only if G is precompact.

In virtue of Theorem (3.5.3) and a (3.5.4), the following two properties are equivalent for every topological group G :

1. The canonical map $G \to W(G)$ is injective.
2. There exists a faithful representation of G by isometries of reflexive Banach space.

Similarly, the canonical map $G \to W(G)$ is homeomorphic embedding if and only if G is isomorphic to a topological sub group of $Is(X)$ for some reflexive Banach space.

Theorem (3.5.5): Let $G = H_+(I)$ be the group of all orientation preserving homeomorphisms of $I = [0,1]$. Then W(G) is a singleton. Equivalently, every $w.a.p.$ function on G is constant.

3.6 The group $Is(U)$

Definition (3.6.1): The group's of Urysohn universal metric space U .

A metric space M is w-homogeneous if every isometry between two finite subsets of M extends to an isometry of M into itself. A metric space M is finitely injective f it has the following property: If K a finite metric space and $L \subset K$, then every isometric embedding $K \to M$. The Urysohn universal space U is the unique complete separable metric space with the following properties: (1) U contains an isometric copy of any separable metric space; (2) U is w-homogenous. Equivalently, U is the unique finitely-injective complete separable metric space. The uniqueness of U is easy: Given two separable finitely injective spaces U_1 and U_2 , one can use the "back and forth" or "shuttle" method to construct an isometry between countable dense subsets of U_1 and U_2 . If U_1 and U_2 are complete, they are isometric themselves.

Let $G = Is(U)$. The group G is a universal topological group with a countable base; every topological group H with a countable base is isometric to a subgroup of G . The idea of the proof is first to embed G into $Is(M)$ for separable metric space M and then to embed M into U in such a way that every isometry of M has a natural extension to an isometry of U .

Let (X, d) be a metric space. We say that a function $f : X \to R_+$ is Katetov if: $|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$ for all $x, y \in X$. A function f is katetov if and only if there exists a metric space $Y = X \cup \{p\}$ containing X as a subspace such that $f(x)$ for every $x \in X$ is equal to the distance between x and p . Let $E(X)$ be the set of all Katetov functions on X , equipped with the sub-metric. If Y is a non-empty subset of X and $f \in E(Y)$, define $g = K_Y(f) \in E(X)$ by

$$g(x) = inf\{d(x, y) + f(y) : y \in Y\}$$

for every $x \in X$. It is easy to check that g is indeed a Katetov function on X and that g extends f . The map $K_r : E(Y) \to E(X)$ is an isometric embedding. Let $X^* = U\{K_r(E(Y): Y \subset X, Y\ ) \; is \; finite \; and \; non-empty\} \subset E(X)$.

For every $x \in X$ let $h_x \in E(X)$ be the function on X defined by $h_x(y) = d(x, y)$. Note that $h_x = K_{\{x\}}(0)$ and hence $h_x \in X^*$. The map $x \to h_x$ is an isometric embedding of X into X^* Thus we can identify X with a subspace of X^* . If K is a finite metric space, $L \subset$

K and $|K/L| = 1$, then every isometric embedding of L into X can be extended to an isometric embedding of L into X^* .

Every isometry of X has a canonical extension to isometry of X^* , and we get an embedding of topological group $Is(X) \rightarrow Is(X^*)$. (Note that the natural homeomorphism $Is(X) \rightarrow Is(E(X))$ in general need not be continuous). Iterating the construction of X^* , we get an increasing sequence of metric spaces $X \subset X^* \subset X^{**} \cdots \cdots$. Let Y be the union of this sequence, and let $\bar{Y}$ be the completion of Y . We have a sequence of embedding of topological group

$$Is(X) \rightarrow Is(X^*) \rightarrow E(X^{**}) \rightarrow E(Y) \rightarrow E(\bar{Y})$$

The space Y is finitely-injective. $\bar{Y}$ the completion of a finitely-injective. Assume that X is separable, then Y is separable, and $\bar{Y}$ is a complete separable finitely-injective metric space. Thus $\bar{Y}$is isometric to U , and hence $Is(X)$ is isomorphic to to a topological subgroup of $Is(U)$.

Every topological group G with a countable base is isomorphic to subgroup of $Is(X)$ for some separable Banach space X : There is a countable subset$A \in R \cup C^b(G)$ which generates the topology of G , and we can take for X the closed G-invarient linear subspace of $R \cup C^b(G)$ generated by A . We just saw that $Is(X)$ isomorphic to a subgroup of $Is(U)$. Thus, we have proved:

Theorem (3.6.2): Every topological group with a countable base is isomorphic to a topological subgroup of the group$Is(U)$.

Note that the group $Is(U)$ is Polish (= separable completely metrizable). Another example of a universal Polish group is the group$H(G)$ of all homeomorphisms of the Hilbert cube. To prove that every topological group$G$ with a countable base is isomorphic to a subgroup of $H(Q)$, it suffices to observe that:

1. G is isomorphic to a subgroup of $H(K)$ for some metrizable compact space K.
2. if K is compact and P(K) is the compact space of all probability measures on $K,$ there is a natural embedding of topological groups $H(K) \rightarrow H\big(P(K)\big)$;
3. If $K$is an infinite separable metrizable compact space, then $P(K)$ is homeomorphic to the Hilbert cube. The groups $Is(U)$and $H(Q)$ are not isomorphic and the group $H(Q)$ is not extremely amenable, since the natural of $H(Q)$ on Q has no fixed point.

The group $Is(U)$ is not Roelcke-precompact, to see that, fix $a \in U$ and consider the function $g \to d(a, g(a))$ from $Is(U)$ to R , where d is the metric on U . This function $L\wedge R$ uniformly continuous and unbounded, hence the Roelcke uniformity$L\wedge R$ is not precompact. We slightly modify the space U , in order to obtain a Roelcke –precompact group of isometries.

Let U_1be the "Urysohn universal metric space in the of spaces of diameter ≤ 1" . This space is characterized by the following properties:U_1 is a complete separable $w-$homogenous metric space of diameter$\leq 1$is isometric to a sub-space ofU_1 . Let $G = Is(U_1)$. This is a universal Polish group. This group is Roelcke precompact. Let us describe the Roelcke compactifications $R(G)$ of G .

Consider the compact space $K \subset I^{U_1}$ of all non-expanding functions $f: U_1 \to I = [0,1]$. Then K is a $G-$space, so there is a natural map from $R(G)$ to the set $ExpK^2$ of all closed relations on$K$. It turns out this map is homeomorphic embedding.

There is a more geometric description of $R(G)$; it is the space of all metric spaces M of diameter 1, which are covered by two isometric copies of U_1 . More precisely, consider all triples $S = (M, i, j)$, where M is a metric spaces M of diameter 1 , $i : U_1 \to M$ and $j : U_1 \to M$ are isometric embeddings, and $M = i(U_1) \cup j(U_1)$. Every such triples gives rise to the function $P_s : U_1 \times U_1 \to I$ defined by $P_s(x, y = d(i(x), j(y))$ where d is the metric on M. The set θ of all functions P , that arise in this way, is a compact subspace of $I^{{U_1}^2}$, and $R(G)$ can be identified with θ. Elements of G correspond to triples (M, i, j) such that $M = i(U_1) = j(U_1)$.

The space $R(G)$ has a natural structure of an ordered semi group. If $R(G)$ is identified with a subset of $ExpK^2$, then $R(G)$ happens to be closed under composition of relations, whence the semi group structure, and the order is just the inclusion. If $R(G)$ is identified with θ , then the order is again natural, and the semi group operation is defined as follows: If $p, q \in \theta$, the product of p and q in θ is the function $r : {U_1}^2 \to I$ defined by

$r(x, y) = inf(\{p(x, z) + q(z, y)\}) \cup \{I\} , x, y \in U_1$.

There is a one-to one correspondence between idempotents in $R(G)$ and closed subsets of U_1 .

Theorem (3.6.3): The universal Polish group $Is(U)$ is minimal.

Thus, every topological group with countable base is isomorphic to a subgroup of a minimal Roelcke-precompact Polish group. More generally, every topological group is isomorphic to a subgroup of a minimal group of the same weight.

Every topological group G is isomorphic to a subgroup of $Is(X)$ **,** where$X$ is a complete w-homogenous metric space of diameter 1, which is injective with respect to finite metric spaces of diameter 1, and for every such X the group $Is(U)$ is Roelcke-precompact and minimal. The uniqueness of X is lost in the one separable case, and it is not known whether there exists a universal topological group of a given uncountable weight.

4 Conclusion

We saw that the topological groups divide to many parts, like abelian and non-abelian topological groups and topological transformation groups, and plays a very important part in pure mathematics. There are many parts of topological groups' compactifications, and they may be useful for studying the group G itself.

References

Arther A. Sagle; Ralph E. Walde (1973). *Lie groups and Lie algebras.* 5th edition. Academic Press, New York.

Artur Kornilowicz (1988). *The definition and basic properties of topological groups*. (TOPGRP!), htt://www. Mizar org/JEM/. Vol. 8, (No 2), pp. 1-12.

Korner T.W. (2003). *Topological groups*. part III. pp. 5-9.

Valdimir Uspenskli, (2001). *Compactifications of topological groups. Proceedings of the Ninth Prague Topological Symposium*. Prague. Czech Republic. pp. 331-346.

Ahmed M. (2004). *Topics in topological groups with some applications of category theory in topological abelian groups*. 1st edition. Khartoum, Sudan.