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On the Boundary Value Problems of Hadamard Fractional Differential EquationswithGeneralized Ulam-HyersStability in BanachSpaces

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Abstract:

Weshow the mild solutions in Banach space for a first order semi-linear integro-differential equation. We discuss a generalized quantum fractional Sturm-Liouville-Langevin difference problem with terminal boundary conditions. We characterize the relevant results rely on Mönch's fixed point theorem by terms of Kuratowski measure of non-compactness (MNC) and the Banach contraction principle (BCP). We study the existence and the stability of solutions of the boundary value problems of Hadamard-type fractional differential equations of variable order.
***Keywords:** derivatives and integrals of variable-order; boundary value problem(BVP); Darbo's fixed point; noncompactness; Ulam-Hyers-Rassias stability; Hadamard derivative.*

حول مسائل القيمة الحدية لمعادلات هادامارد التفاضلية الكسرية مع استقرار أولام-هايرز المعمم في فضاءات باناخ

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المستخلص

أوضحنا الحلول المعتدلة في فضاء باناخ لمعادلة التفاضلية التكاملية شبه الخطية من الدرجة الأولى. ناقشنا مسألة الاختلاف الكمي الجزئي المعمم لستورمليوفيل–لانجفين مع شروط الحدود النهائية. شخصنا النتائج ذات الصلة المعتمدة على نظرية مونش للنقطة الثابتة من خلال مقياس كوراتوفسكي لعدم الانضغاط (MNC) ومبدأ انكماش باناخ(BCP). درسنا وجود استقرارية حلول مشاكل القيمة الحدودية للمعادلات التفاضلية الكسرية من نوع هادامارد ذات الرتبةالمتغيرة.

***الكلمات المفتاحية:** المشتقات والتكاملات ذات الترتيب المتغير، مشكلة القيمة الحدودية،النقطة الثابتة لداربو، قياس عدم الاكتناز، استقرار أولام-هايرز-راسياس، مشتق هادمارد.*

Introduction:

Integro-differential equations appear in many fields of sciences such as physics, biology and other fields of applied mathematics. This kind of equations has received considerable attention. Many researchers used the measure of non-compactness to study the existence of mild solution for various forms of integral equations, differential equations, and integro-differential equations. The topics of fractional calculus and (quantum) q-calculus in general and fractional differential equations, especially, have appeared extensively, and they are one of the applied branches in mathematical analysis which have enormous impact in exact description of existing real phenomena. In the meantime, in 1910, q-difference equations were introduced by Jackson. The idea of fractional calculus is to replace the natural numbers in the derivative's order with rational ones. Although it seems an elementary consideration, it has an exciting correspondence explaining some physical phenomena. Studying both of the theoretical and practical aspects of fractional differential equations (FDEqs) has become a focus of an extensive international academic research. We derive sufficient conditions for the existence of solution of the following class of integro-differential equation

$$(1)\qquad \begin{cases} dL(t) = [OL(t) + \int_0^t \Gamma(t-s)L(s)ds + g(t, L(t), \int_0^t h(t,s,L(s))ds)]dt, \\ t \in I = [0,T], \\ L(0) = \iota_0, \end{cases}$$

where $(X,|\cdot|)$is a real Banach space, □is the infinitesimal generator of a strongly continuous semi-group of bounded linear operators $(S(t))_{t\geq 0}$ on X and □(t) is a linear closed operator on X with domain $D(O)\subset D(□)$.

The function h: $I \times I \times X \to X$, with $s \leq t$ for $(s,t)\in I \times I$ is continuous in X,

g: $I \times X \times X \to X$ is a Caratheodory function.

We will investigate the existence of solutions for the previously mentioned integro-differential system since this problem still has not been considered in the literature. Our contributions are: a general class of integro-differential equation is considered in this work. Then, using functional analysis methods, a set of sufficient conditions is proposed to ensure the existence of mild solutions. The results are established through the use of the theory of the resolvent operator in the Grimmer sense. We use the fact that the operator-norm continuity of the resolvent operator is equivalent to that of the semi-group. This property allows us to drop the supposition that the operator semi-group is compact. This is an interesting result in itself and is of key practical importance. Throughout this section $(X,|\cdot|)$ is a

Banach space and C(I,X) denote the Banach space of continuous functions from I into X furnished by the usual supremum norm $\| L \| = \sup_{t\in I} | L(t) |, \quad \forall L \in C(I, X).$

Next to be able to reach existence of mild solutions for (1), we recall some details on partial integro-differential equations and resolvent operators that will be used to establish the key results.
We look at the following Cauchy problem

$$(2) \qquad \begin{cases} x'(t) = Ox(t) + \int_0^t \Gamma(t-s)x(s)ds \quad \text{for } t \geq 0, \\ x(0) = x_0 \in X, \end{cases}$$

where X is a Banach space, O and □(t) are closed linear operators on X with domains D(O)⊂D(□). Y represents the Banach space D(O) furnished with the graph norm defined by $|y|_Y := |O_y|+|y|$ for y∈Y. We denote by C([0,+∞); Y), and B(Y, X), the space of all continuous functions from [0,+∞) into Y, the set of all bounded linear operators from Y into X, respectively. If X = Y, we write B(X) instead of B(X,X).
In particular, Agarwal *et al.* (2007)studied the following problem:

$$\begin{cases} D_{0+}^{u} x(t) = f(t, x(t)), \quad t \in I := [0, \infty), \quad u \in]1,2], \\ x(0) = 0, \qquad x \text{ bound on } [0, \infty), \end{cases}$$

where D_{0+}^{u} is the Riemann-Liouville fractional derivative of order u, f is a given function.We deal with the following boundary value problem (BVP)

$$(3) \qquad \begin{cases} {}^{H}D_{1+}^{u(t)} x(t) = f_1(t, x(t)), \quad t \in J := [1, T], \\ x(1) = x(T) = 0, \end{cases}$$

where $1 < u(t) \leq 2$, f_1: J×X →X is a continuous function and ${}^{H}D_{1+}^{u(t)}$, ${}^{H}I_{1+}^{u(t)}$, are the Hadamard fractional derivative and integral of variable-order u(t).The formal definitions and properties of the Hadamard fractional derivatives andintegrals of variable-order.The goal of our research is to propose new existence criteria for the solutions of (3). In addition, we study the stability of the obtained solution of (3) in the sense of Ulam-Hyers-Rassias (UHR)(Karapinar*et al*., 2021).

Definition(1) We call resolvent operator for the problem (2), a bounded linear operator valued function R(t)∈B(X) for t ≥ 0, having the following properties:(Mariam *et al*., 2021).

a) R(0)=I (identity operator on X) and ‖R(t)‖≤ $Me^{\square t}$ for some constants M and□□;

b) for each x∈X, R(t)x is strongly continuous for t ≥ 0;

c) R(t)∈B(Y) for t ≥ 0, for x∈Y, R(·) x∈C^1([0,+∞),X) ∩ C([0,+∞), Y) and

$$R'(t) = OR(t)x + \int_0^t \Gamma(t-s)R(s)ds,, = R(t)Ox + \int_0^t R(t-s)\Gamma(s)xds, \text{for } t \geq 0.$$

Remark(2) The above example shows that in general, the resolvent operator $R(t)_{t\geq 0}$ for equation (2) does not satisfy the semi-group law, namely,

$$R(t+s) \neq R(t)R(s) \qquad \text{for some } t, s > 0.$$

The existence of resolvent operator has been discussed. In what follows, we suppose the following assumptions (Mariam *et al*., 2021).

(H_1) O is the infinitesimal generator of strongly continuous semi-group $(R(t))_{t\geq 0}$ on X.

(H_2) For all $t \geq 0$, $\Gamma(t)$ is closed operator from D(O) to X and $\Gamma(t) \in B(Y,X)$. For any $y \in Y$, the map $t \mapsto \Gamma(t)y$ is bounded, differentiable and the derivative

$t \mapsto \Gamma'(t)y$ is bounded uniformly continuous on R^+.

Definition(3) We call mild solution of the system (1), a function $L \in C(I,X)$ for which the following integral equation is satisfied

$$L(t) = R(t)L_0 + \int_0^t R(t-s)\, g(s, L(s), \int_0^S h(s,\xi,L(\xi))d\xi)\, ds \cdot$$

In order to prove our results, we introduce the following assumptions.

(H_3) There is a continuous function u: $I \to R^+$, such that

$$|g(t,x,y)| \leq u(t)(|x|+|y|) \text{ for all } x,y \in X \text{ and a.e. } t \in I.$$

(H_4) There are integrable functions $\rho_i(i = 1,2)$: $I \to R^+$, such that:

$$\nu(g(t,O_1,O_2)) \leq \rho_1(t)\nu(O_1) + \rho_2(t_2)\nu(O_2), \text{for a.e. } t \in I \text{ and } O_1,O_2 \subset X.$$

(H_5) There is a continuous function ν: $I \to R^+$, such that:

$$|h(t,s,x)| \in \nu(t)|x|, \text{ for a.e. } (t,s) \in f(t,s) \in I \times I;\ s \leq t\} \text{ and } x \in X.$$

(H_6) There is a constant $h^* > 0$, such that:

$$\nu(h(t,s,D)) \leq h^*\nu(D), \text{for a.e. } (t,s) \in J = f(t,s) \in I \times I;\ s \leq t\} \text{ and } D \in X.$$

(Mariam *et al*., 2021)

Definition (4) Let B be a bounded subset of a Banach space X. The Kuratowski measure of non-compactnessν is defined by

$\nu(B) = \inf\{\varepsilon > 0$: B has a finite cover by sets of diameter smaller than $\varepsilon\}$.

(Mariam et al., 2021)

Lemma (5) Let X be a Banach space and $B,C \subset X$ be bounded, then the following properties are satisfied:(Mariam *et al*., 2021)

i. $\nu(B) = 0$ if only if B is relatively compact;

ii. $\nu(\overline{B}) = \nu(B)$; $\overline{B}$ the closure of B;

iii. $\nu(C) < \nu(B)$, when $C \subset B$;

iv. $\nu(C+B) < \nu(C) + \nu(B)$, where $C+B = \{d \mid d = b+c;\ b\in B,\ c\in C\}$;

v. $\nu(rB) = |r|\nu(B)$ for all $r\in R$;

vi. $\nu(Conv(B)) = \nu(B)$, where Conv(B) is the convex hull of B;

vii. $\nu(C\cup B) = \max(\nu(C), \nu(B))$;

viii. $\nu(C\cup\{x\}) = \nu(C)$ for all $x\in X$.

Denote by $\eta^T(y,\varepsilon)$ the modulus of continuity of y on the interval [0,T], i.e.,

$$\eta^T(L,\varepsilon) = \sup\{|L(t) - L(s)|;\ t, s \in I,\ |t-s| \le \varepsilon\}.$$

We define, $\eta^T(B,\varepsilon) = \sup\{\eta^T(L,\varepsilon), L \in B\}, \eta_0^T = \lim_{\varepsilon\to 0}\eta^T(B,\varepsilon).$

Lemma(6) Let X be a Banach space, B be a bounded subset of X. Then there is a countable and bounded set $B_0 \subset B$ such that $\nu(B) \le 2\nu(B_0)$.

(Mariam *et al.*, 2021)

Lemma(7) Let X be a Banach space, B $=\{\iota_n\}_{n=0}^{\infty} \subset C(I,X)$ be a bounded and countable set. Then $\nu(B(t))$ is Lebesgueintegrable on I, and

$$\nu\left(\{\int_0^t \iota_n(s)ds\}_{n=0}^{\infty}\right) \le 2\int_0^t \nu\,(B(s))ds,\ t \in I.$$

(Mariam *et al.*, 2021)

Lemma(8) Let w(t),u(t) and $\nu(t)$ be real valued nonnegative continuous function defined on R^+, for which the inequality

$$w(t) \le w_0 + \int_0^t u(s)[w(s) + \int_0^t \nu(\xi)w(\xi)d\xi]ds,$$

is satisfied for all $t\in R^+$, where w_0 is a nonegative constant, then

$$w(t) \le w_0[1 + \int_0^t u(s)[\exp(\int_0^s (u(\xi)\nu(\xi))d\xi)]ds\ ], \text{ for all } t\in R^+.$$

Let the subset $W\neq\phi$ be convex and bounded, and in the Banach space U with $0 \in W$, $\sigma: W\to W$ be continuous. If $\forall\Sigma\subset W$,

(4) $\Sigma = \overline{\text{conv}}\,(\Sigma)$ or $\Sigma = \sigma(\Sigma) \cup \{0\} \Rightarrow \kappa(\Sigma) = 0$,

Then σ has a fixed point. (Mariam *et al.*, 2021;Traor´ea, *et al*, 2021)

Lemma(9) Let $K(t)\in U$, $0 < \alpha, \beta \le 1$, $\rho\in C(I,R\backslash\{0\})$, and $r\in C(I,R)$.

Then the solution of the following linear generalized Sturm-Liouville-Langevinq-difference FBVP(Boutiara*et al.*, 2021).

$$(5)\qquad \begin{cases} {}_cD_q^{\alpha}([\rho(t)\ {}_cD_q^{\beta} + r(t)])w(t) = K(t)\ ,\ (t\in I), \\ w(0)=0, \qquad {}_cD_q^{\beta}w(T) + \frac{r(T)}{\rho(T)}w(T) = 0, \end{cases}$$

Is given by

$$(6)\ w(t) = {}_{RL}I_q^{\beta}\left(\frac{1}{\rho}\ {}_{RL}I_q^{\alpha}K\right)(t) - {}_{RL}I_q^{\beta}\left(\frac{r}{\rho}w\right)(t) - {}_{RL}I_q^{\alpha}K(T)\ {}_{RL}I_q^{\beta}\left(\frac{1}{\rho}\right)(t)$$

Proof Taking the αth-q-Riemann-Liouville integral to the FDEq of (5), we get

$$(7)\qquad {}_cD_q^{\beta}w(t) = \frac{{}_{RL}I_q^{\beta}K(t) + c_0 - r(t)w(t)}{\rho(t)}$$

Where $c_0 \in R$. The second BCs of system (5) give

$$c_0 = -\ {}_{RL}I_q^{\alpha}K(T).$$

Taking the β^{th}-q-Riemann-Liouville integral to (7), we obtain

$$(8)\ w(t) = {}_{RL}I_q^{\beta}\left(\frac{1}{\rho}\ {}_{RL}I_q^{\alpha}K\right)(t) - {}_{RL}I_q^{\beta}\left(\frac{r}{\rho}w\right)(t) - {}_{RL}I_q^{\alpha}K(T)\ {}_{RL}I_q^{\beta}\left(\frac{1}{\rho}\right)(t) + c_1.$$

Where, $c_1 \in R$. Using the condition w(0) = 0 of (5), we have $c_1 = 0$.

Substituting the obtained value for c_1, we derive the q-integral equation (6).

Now, consider the nonlinear generalized Sturm-Liouville-Langevin q-difference FBVP (4). On the basis of Lemma (1.9), the solutions of (5) correspond to q-integral equation in the following form:

$$(9)\qquad \begin{aligned} w(t) &= {}_{RL}I_q^{\beta}\left(\frac{1}{\rho}\ {}_{RL}I^{\alpha}\sigma\right)(t, w(t)) - {}_{RL}I^{\beta}\left(\frac{r}{\rho}w\right)(t) \\ &- {}_{RL}I^{\alpha}\sigma(T, w(T))\ {}_{RL}I_q^{\beta}\left(\frac{1}{\rho}\right)(t) \end{aligned}$$

We further will use the following hypotheses.

(H_1) $\sigma: I\times U \to U$ is Caratheodory;

(H_2) There exists $p\in C(I,R^+)$ such that $\|\sigma(t, w(t))\| \le p(t)\|w\|$, $\forall t\in I$, $\forall w\in U$;

(H_3) For each $t\in I$ and each bounded measurable set $B \subset U$,

$$\lim_{h\to 0^+} \kappa(\sigma(I_{t,h}\times B),0) \le p(t)\kappa(B),$$

Where, κ is the Kuratowski MNC and $I_{t,h} = [t-h, t] \cap I$.

Set

$$(10)\qquad p^* = \sup_{t\in I}|p(t)|, \qquad \rho^* = \inf_{t\in I}|\rho(t)|, \qquad r^* = \sup_{t\in I}|r(t)|$$

Existence result via the KMNC-method

Theorem(1) Assume that (H_1)-(H_2) hold. Then there exists a unique resolvent operator of the Cauchy problem (2). We have the following theorem that establishes the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations. Let O be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and let $(\Gamma(t))_{t\geq 0}$ satisfies (H_2). Then the resolvent operator $(R(t))_{t\geq 0}$ for equation (2) is operator-norm continuous (or continuous in the uniform operator topology) for t > 0 if and only if $(T(t))_{t\geq 0}$ is operator-norm continuous for t>0(Mariam *et al*., 2021).

Theorem(2) Let E be a Banach space, Ω be a bounded open subset of E and 0∈Ω. Suppose that F: $\overline{\Omega}$ →E is ν-condensing and assume thatu ≠λF(u) for u∈∂ and λ∈(0,1) hold. Then F has a fixed point in $\overline{\Omega}$(Mariam *et al.,* 2021).

Theorem(3) Assume that (H_1)-(H_2) hold. Then there exists a unique resolvent operator of the Cauchy problem (2). We have the following theorem that establishes the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations. Let O be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and let $(\square(t))_{t\geq 0}$ satisfies (H_2). Then the resolvent operator $(R(t))_{t\geq 0}$ for equation (2) is operator-norm continuous (or continuous in the uniform operator topology) for t > 0 if and only if $(T(t))_{t\geq 0}$ is operator-norm continuous fort > 0(Mariam *et al*., 2021).

Theorem(4) Assume that equation(2) has a resolvent operator$(R(t))_{t\geq 0}$ which is continuous in the operator-norm topology for t > 0 and hypotheses (H_1)-(H_6) hold. Then there exists at least one mild solution of (1).

Proof. Let Q: C(J,X)→C(J,X), be a operator defined by

$$(QL)(t) = R(t)L_0 + \int_0^t R(t-s)g(s, L(s), \int_0^s h(s,\xi,L(\xi))\,d\xi)\,ds \cdot$$

A mild solution of the system (1) is a fixed point of the operator Q. Let P be a positive constant. We consider the set B_P={L∈C(J,X): ||L|| < P}. The proof will be splited into the following steps(Mariam *et al*., 2021).

Step 1. Q(B_P) is bounded for any bounded set B_P. For L∈B_P, and t∈I, we have

$$|QL|(t) \leq \|R(t)\|\,|L_0| + \int_0^t \|R(t-s)\|\, g(s, L(s),\ ds\int_0^s h(s,\xi,L(\xi))d\xi)\,ds$$

$$\leq N|L_0| + N\int_0^t u(s)(|L(s)| + \int_0^s \nu(s,|L(\xi)|)d\xi)\,ds \leq N|L_0|$$

$$+ NPT\|u\|\left(1+\frac{T}{2}\|\nu\|\right) < \infty$$

Thus Q(B_P) is bounded for all L∈B_P.

Step 2. Q is continuous. To seek this, let $(\iota_n)_{n\in N}$ be a convergent sequence in B_P such that $\iota_n \to \iota$. Then we have, for all $t\in I$

$$\|(Q\iota_n)(t)-(Q\iota)(t)\|$$
$$\leq N\int_0^t |g\ (s,\iota_n(s),\int_0^s h(s,\xi,\iota_n(\xi))d\xi)-g(s,\iota(s),\int_0^t h(s,\xi,\iota(\xi))d\xi)|\,ds$$

Since h is continuous and g is a Caratheodory function, using the Lebesgue dominated convergence theorem, we obtain $\|Q\iota_n - Q\iota\| \to 0$ as $n\to+\infty$,

which implies that Q is a continuous operator on B_P.

Step 3. $Q(B_p)$ is equicontinuous. Let $t_1,t_2\in I$ with $t_1< t_2$ and $L\in B_P$. We have:

$$|(QL)(t_2)-(QL)(t_1)|\leq|\int_0^{t_1}(R(t_2-s)-R(t_1-s))g(s,L(s),\int_0^s h(s,\xi,L(\xi))d\xi)\,ds|$$
$$+|\int_{t_1}^{t_2} R(t_2-s)g(s,L(s),\int_0^s h(s,\xi,L(\xi))d\xi)\,ds|$$

We obtain

$$\leq\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|_{B(X)}\,u(s)(|L(s)|+\int_0^s \nu(\xi)|L(\xi)|d\xi)ds.$$
$$+N\int_{t_1}^{t_2} u(s)(|L(s)|+\int_0^s \nu(\xi)|L(\xi)|d\xi)ds.$$

$$\|(QL)(t_2)-(QL)(t_1)\|\leq p\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|_{B(x)}\,u(s)(1+\int_0^s \nu(\xi)d\xi)\,ds$$
$$+NP\int_{t_1}^{t_2}\sup_{t\in I}|u(t)|(1+\int_0^s \sup_{t\in I}|\nu(t)|d\xi)ds$$
$$\leq P\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|_{B(X)}\,u(s)(1+\int_0^s \nu(\xi)d\xi)\,ds+NP\|u\|\int_{t_1}^{t_2}(1+s\|\nu\|)ds$$
$$\leq P\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|_{B(X)}\,u(s)(1+\int_0^s \nu(\xi)d\xi)ds$$
$$+NP\|u\|((t_2-t_1)+\frac{1}{2}\|\nu\|(t_2-t_1)^2).$$

By Theorem (2.3), the right-hand side of the above inequality tends to zero as $t_2 - t_1\to 0$. Consequently $Q(B_P)$ is equi-continuous.

Setp 4. We show that Q is condensing operator. Let $\nu^*(D)$ be the measure of non-compactness defined on the family of bounded subsets of the space

C(I, X) by, $$\nu^*(D)=\eta_0^T(D)+\sup_{t\in I} e^{-\xi\rho(t)}\bar{\nu}(D(t)),$$

where, $$\rho(t)=4N\int_0^t(\rho_1(s)+2h^*s\rho_2(s))ds,\ \xi\geq 1,\ \bar{\nu}(D(t))=\sup_{s\in[0,t]}\nu(D(s)).$$

For all $D \subset B_P$, Q(D) is bounded. Therefore, a countable set $D_0 = \{\iota_n\}_{n=1}^{\infty} \subset D$ exists by Lemma (1.5) so that

(11) $\nu(Q(D)(t)) \leq 2\nu(Q(D_0)(t))$ for each $t \in I$.

By the properties of ν, hypotheses (H_3),(H_4),(H_5), and Lemmas (1.5) and (1.6), we obtain

$$\nu|Q(D_0)(t)| \leq \nu\left(\{\int_0^t R(t-s)g(s,\iota_n(s),\int_0^s h(s,\xi,\iota_n(\xi))d\xi)ds\}_{n=0}^{\infty}\right)$$

$$\leq 2N\int_0^t \{\nu(g(s,\iota_n(s),\int_0^s h(s,\xi,\iota_n(\xi))d\xi)\}_{n=0}^{\infty} ds$$

$$\leq 2N\int_0^t \left(\rho_1(s)\{\nu(\iota_n(s))\}_{n=0}^{\infty} ds + \rho_2(s)\{\nu(\int_0^t h(s,\xi,\iota_n(\xi))d\xi)\}_{n=0}^{\infty}\right)ds$$

$$\leq 2N\int_0^t \left(\rho_1(s)\{\nu(\iota_n(s))\}_{n=0}^{\infty} + 2h^*\rho_2(s)\{(\int_0^s \nu(\xi,\iota_n(\xi))d\xi)\}_{n=0}^{\infty}\right)ds$$

$$\leq 2N\int_0^t \left(\rho_1(s)\nu(D_0(s)) + 2h^*\rho_2(s)\int_0^s \nu(D_0(\xi))d\xi\right)ds$$

$$\leq 2N\int_0^t (\rho_1(s)\nu(D_0(s)) + 2h^*\rho_2(s)s \sup_{\xi\in[0,s]} \nu(D_0(\xi)))ds$$

$$\leq 2N\int_0^t (\rho_1(s) \sup_{s\in[0,t]} \nu(D_0(s)) + 2h^*\rho_2(s)s \sup_{\xi\in[0,t]} \nu(D_0(\xi)))ds$$

$$\leq 2N\int_0^t (\rho_1(s) + 2h^* s\rho_2(s)) \sup_{s\in[0,t]} \nu(D_0(s))ds$$

From inequality (11), it follows that

$$\nu(QD(t)) \leq 4N\int_0^t (\rho_1(s) + 2h^*\rho_2(s)s)\overline{\nu}(D(s))ds.$$

We know $\nu(QD(t)) \leq 4N \int_0^t (\rho_1(s) + 2h^* s\rho_2(s))e^{\xi,\rho(s)} e^{-\xi,\rho(s)} \overline{\nu}(D(s))ds$

Then, $e^{-\xi,\rho(t)} \nu(QD(t)) \leq \frac{1}{\xi} \sup_{t\in I} e^{-\xi,\rho(t)} \overline{\nu}(D(t))$

Hence, $e^{-\xi,\rho(t)} \sup_{t\in I} \nu(QD(t)) \leq \frac{1}{\xi} \sup_{t\in I} e^{-\xi,\rho(t)} \overline{\nu}(D(t))$

Since $e^{-\xi,\rho(t)} \sup_{s\in[0,t]} \nu(QD(s)) \leq e^{-\xi,\rho(t)} \sup_{t\in I} \overline{\nu}(QD(t))$,

We obtain, $e^{-\xi,\rho(t)} \sup_{s\in[0,t]} \nu(QD(s)) \leq \frac{1}{\xi} \sup_{t\in I} e^{-\xi,\rho(t)} \overline{\nu}(D(t))$.

Then

(12) $$e^{-\xi,\rho(t)}\,\overline{\nu}(QD(t)) \le \frac{1}{\xi}\sup_{t\in I} e^{-\xi,\rho(t)}\,\overline{\nu}(D(t)).$$

From Step 3 and inequality (12), we obtain $\nu^*(QD) \le \nu^*(D)$.Consequently, Q is condensing.

Step 5. We show that there is an open set $O \subseteq C(I,X)$ with $L \neq \beta Q(L)$, for $\beta\in(0,1)$ and $L\in\partial O$. Let $L\in C(I,X)$, suppose that $L = \beta Q(L)$ for $\beta\in(0,1)$. Then

$$L(t) \le \beta R(t)L_0 + \beta\int_0^t R(t-s)g(s,L(s),\int_0^s h(s,\xi,L(\xi))d\xi)\,ds\ .$$

By (H_1)-(H_3) and (H_5), for each $t\in I$, we have

$$|L(t)| \le \| R(t)\|_{B(X)}\ |L_0| + \int_0^t \| R(t-s)\|_{B(X)}\, u(s)(|L(s)| + \int_0^s \nu(\xi)\,|L(\xi)|\,d\xi)ds$$

$$\le N\,|L_0| + N\int_0^t u(s)(|L(s)| + \int_0^t \nu(\xi)\,|L(\xi)|\,d\xi)ds$$

By Lemma (1.8), we get

$$|L(t)| \le N\,|L_0|\left(1 + \int_0^t Nu(s)\exp(\int_0^s (Nu(\xi)+\nu(\xi))\,d\xi)\,ds\right)$$

$$\le N\,|L_0|\left(1 + N\int_0^t \sup_{t\in I}|u(t)|\exp(\int_0^s (N\sup_{t\in I}|u(t)| + \sup_{t\in I}|\nu(t)|)\,d\xi)\,ds\right)$$

$$\le N\,|L_0|(1 + NT\|u\|\exp(T(N\|u\| + \|\nu\|))) = \tau$$

Thus, we get $\|L\| \le \tau$. We consider the set, $O = \{L\in C(I,X): \|L\| < \tau + 1\}$.

By the choice of O, there is no $L\in\partial O$ thus $L= \beta Q(L)$, for $\beta\in(0,1)$. Thus by Theorem (2.2), we conclude that Q has a fixed point that is a solution of the system (1).

Theorem(2.5) Let L be nonempty, closed, bounded and convex subset ofa Banach space X and W: $\Lambda\to\Lambda$ a continuous operator satisfying

$$\zeta(W(S)) \le k\zeta(S) \text{ for any } (S \neq \phi) \subset \Lambda,\ k\in[0, 1).$$

Then, W has at least one fixed point in Λ.(Reficee*et al.*, 2021)

Some Examples

Example(1) Let

$$\nu(t) = \begin{cases} 1, t\in[1,2] \\ 2, t\in]2,4], \end{cases} \qquad u(t) = \begin{cases} 2, t\in[1,2] \\ 1,\ t\in]2,4], \end{cases} \quad f_2(t) = 1,\ \ t\in[1,4],$$

$$ {}^{H}I_{1^+}^{u(t)}({}^{H}I_{1^+}^{v(t)})f_2(t) = $$

$$ \frac{1}{\Gamma(2)}\int_1^2 \frac{1}{s}(\log\frac{t}{s})^1\left[\frac{1}{\Gamma(1)}\int_1^2 \frac{1}{\tau}(\log\frac{s}{\tau})^{1-1}d\tau + \frac{1}{\Gamma(2)}\int_2^s \frac{1}{\tau}(\log\frac{s}{\tau})^{2-1}d\tau\right] ds $$

$$ + \frac{1}{\Gamma(1)}\int_2^t \frac{1}{s}(\log\frac{t}{s})^{1-1}\left[\frac{1}{\Gamma(1)}\int_1^2 \frac{1}{\tau}(\log\frac{s}{\tau})^{1-1}d\tau + \frac{1}{\Gamma(2)}\int_2^s \frac{1}{\tau}(\log\frac{s}{\tau})^{2-1}d\tau\right] ds $$

and, $ {}^{H}I_{1^+}^{u(t)+v(t)} f_2(t) = \frac{1}{\Gamma(u(t)+v(t))}\int_1^t \frac{1}{s}(\log\frac{t}{s})^{u(t)+v(t)-1} f_2(s)ds $

Thus, we get

$$ {}^{H}I_{1^+}^{u(t)}({}^{H}I_{1^+}^{v(t)})f_2(t)\,|_{t=3} = $$

$$ \frac{1}{\Gamma(2)}\int_1^2 \frac{1}{s}(\log\frac{3}{s})^1\left[\log 2 + \frac{1}{s}(\log\frac{s}{2})^2\right] ds + \frac{1}{\Gamma(1)}\int_2^3 \frac{1}{s}\left[\log 2 + \frac{1}{2}(\log\frac{s}{2})^2\right] ds \approx 0.9013 $$

$$ {}^{H}I_{1^+}^{u(t)+v(t)} f_2(t)\,|_{t=3} = \frac{1}{\Gamma(3)}\int_1^2 \frac{1}{s}(\log\frac{2}{s})^2 ds + \frac{1}{\Gamma(3)}\int_2^3 \frac{1}{s}(\log\frac{3}{s})^2 ds \qquad \approx 0.2209 $$

We obtain $ {}^{H}I_{1^+}^{u(t)}({}^{H}I_{1^+}^{v(t)})f_2(t)\,|_{t=3} \neq {}^{H}I_{1^+}^{u(t)+v(t)} f_2(t)\,|_{t=3} $.

(Ahmed Reficeet *et al.*, 2021)

Example(2) We consider the following semi-linear-integro-differential Equation(Mariam et al., 2021)

$$ \frac{\partial}{\partial t}x(t,\xi) = \left[\frac{\partial^2 x(t,\xi)}{\partial \xi^2} + \breve{b}\frac{\partial x(t,\xi)}{\partial \xi} + \tilde{c}x(t,\xi)\right] $$

$$ + \int_0^t \Xi(t-s)\left[\frac{\partial^2 x(s,\xi)}{\partial \xi^2} + \breve{b}\frac{\partial x(s,\xi)}{\partial \xi} + \tilde{c}x(s,\xi)\right] ds + \frac{x(t,\xi)}{(\sqrt{t}+1)(1+|x(t,\xi)|)} \tag{13} $$

$$ + \frac{e^{-t}}{(\sqrt{t}+1)(t+1)}\int_0^s \frac{\sqrt{t}x(s,\xi)}{(1+s^2+t)(1+|x^2(s,\xi)|)} ds,\ t\in[0,T],\ \xi\in[0,1], $$

$$ x(t,0) = x(t,1) = 0,\ \text{for } t\in[0,T], $$

$$ x(0,\xi) = x_0,\ t\in[0,T],\in[0,1], $$

where $\Xi : R^+ \mapsto R$ is continuous function, $\tilde{b}, \tilde{c} \in R$.

Let $X = L^2(0,1)$. We define the operator O induced on X as follows:

$$ \begin{cases} D(O) = H^2(0,1) \cap H_0^1(0,1), \\ Oz = z'' + \tilde{b}z' + \tilde{c}z,\ \tilde{b}, \tilde{c} \in R. \end{cases} $$

We know that O is the infinitesimal generator of an analytic C_0semigroup $(S(t))_{t\geq 0}$ on X. Since the semigroup generated by O is analytic, then it is norm continuous for t > 0.

We define the operators $\Gamma(t)$: Y $\mapsto$ X as follows:

$\Gamma(t)N = \Xi(t)ON$ for $t \geq 0$ and $N \in D(O)$.

For every $t \in [0,T]$, we define

$$L(t) = x(t,\zeta)$$

$$g(t,y,z)(\xi) = \frac{1}{(\sqrt{t}+1)(1+|y(t,\xi)|)} y(t,\xi) + \frac{e^{-t}}{(\sqrt{t}+1)(t+1)} z(t,\xi),$$

$$h(t,s,y)(\xi) = \frac{\sqrt{t}y(t,\xi)}{(1+s^2+t)(1+y^2(t,\xi))}.$$

Thus, (13) takes the following abstract form

$$(14)\qquad \begin{cases} dL(t) = [OL(t) + \Gamma\int_0^t \Gamma(t-s)L(s)\,ds + g(t,L(t),\int_0^t h(t,s,L(s)))ds]\,dt, \\ t \in I = [0,T], \\ L(0) = \iota_0. \end{cases}$$

We suppose Ξ is a bounded and C^1 function such that $\Xi'$is bounded and uniformly continuous which implies that the operator $\Gamma(t)$ satisfies (H_2). Thus from Theorem(2.3), problem(14) has a resolvent operator $(R(t))_{t\geq 0}$ on X which is norm continuous for $t > 0$.

We have $$|g(t,y,z)| = \frac{1}{1+\sqrt{t}}(|y(t,\xi)| + |z(t,\xi)|),$$

and

$$(15)\qquad |h(t,s,y)| = \frac{\sqrt{t}}{1+t}|y(t,\xi)|.$$

Hence conditions (H_3) and (H_5) are satisfied with $u(t) = \frac{1}{1+\sqrt{t}}, \quad \nu(t) = \frac{\sqrt{t}}{1+t}.$

From the definition of g, for every $t \in I$, and $B_1, B_2 \in B \subset X$, we have

$$\nu(g(t,B_1,B_2)) \leq \frac{1}{\sqrt{t}+1}\nu(B_1) + \frac{e^{-t}}{(1+\sqrt{3})(1+t)}\nu(B_2).$$

Hence condition (H_4) is satisfied with $\rho_1(t) = \frac{1}{\sqrt{t}+1}$ and $\rho_2 = \frac{e^{-t}}{(1+\sqrt{3})(1+t)}$.

By (15), for every $t \in I$ and $B \subset X$, we have $\nu(h(t,s,B)) \leq \sup_{t\in I}\frac{\sqrt{t}}{t+1}\nu(B)$, then

$$\nu(h(t,s,B)) \leq \frac{\sqrt{2}}{3}\nu(B)$$

Hence (H_6) is satisfied with $h^* = \frac{\sqrt{3}}{2}$. Finally, all assumptions of our main results are satisfied. From Theorem(2.4), we deduce the existence of solution of system(13).

Conclusions

A boundary value problem for Hadamard fractional differential equations of variable order is studied. It leads to the necessity of obtaining existence criteria for a boundary value problem for Hadamard fractional differential equations of variable order. Also, the stability in the sense of Ulam-Hyers-Rassias is investigated. The results are obtained based on the Kuratowski measure of non-compactness. We provided required criteria for the existence/uniqueness of solutions to a new category of nonlinear generalized Sturm-Liouville-Langevin q-difference FBVP. We dealt with a technique involving the Kuratowski measure of noncompactness (KMNC) along with a fixed point theorem ofMönch type. We created generalized Ulam-Hyers (UH) andUlam-Hyers (GUH) stability were established for the proposed nonlinear generalized Sturm-Liouville-Langevin q-difference FBVP.

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