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Connections on the Parallel Vector Fields in Vector Bundles

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Abstract:

This paper is a survey of the basic theory of connection on bundles. A generalised notion of connection on a fibre bundle E over a manifold M is presented. These connections are characterised by a smooth distribution on E which projects onto a (not necessarily integrable) distribution on M and which, in addition, is ‘parametrised’ in some specific way by a vector bundle map from a prescribed vector bundle over M into TM. Some basic properties of these generalised connections are investigated. Special attention is paid to the class of linear connections over a vector bundle map.

Keywords: *Vector bundles; generalised connections; fiber bundles; smooth Vector bundles; parallel sections.*

الاتصال على حقول المتجهات المتوازية على حزم المتجهات

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المستخلص :

هذه الورقة البحثية عبارة عن مسح للنظرية الاساسية للاتصال على حزم الالياف. اظهرنا فكرة عامة عن الاتصال على حزمة الالياف المجمعة. تتميز هذه الاتصالات بالتوزيع المرن على الحزم الذي يظهر على توزيع (ليس بالضرورة قابل للتكامل) متفرع. بالإضافة الى ذلك تم وصف طريقة محددة بواسطة تطبيق حزمة متجه عبر الياف مجمعة. تم دراسة بعض الخصائص الاساسية لهذه الارتباطات العامة . ووضعنا اهتمام خاص لفئة الاتصال عبر خريطة حزمة المتجهات.

الكلمات المفتاحية : *حزم المتجهات، الاتصالات العامة، حزم الياف، حزم المتجهات المرنة، المقاطع المتوازية .*

Introduction

The theory of connections undoubtedly constitutes one of the most beautiful and most important chapters of differential geometry, In order to differentiate sections of a vector bundle or vector fields on a manifold we need to introduce a structure called the connection on a vector bundle. For example, an affine connection is a structure attached to a differentiable manifold so that we can differentiate its tensor fields. We first introduce the general theorem of connections on vector bundles. Then we study the tangent bundle. TMis m-dimensional vector bundle determine intrinsically by the differentiable structure of an◌̃-dimensional smooth manifold M.

Connections on Vector Bundles

A connection on a fiber bundle is a device that defines a notion of parallel transport on the bundle, that is, a way to connect or identify fibers over nearby points. If the fiber bundle is a vector bundle, then the notion of parallel transport is required to be linear. Such a connection is equivalently specified by a covariant derivative, which is an operator that can differentiate sections of that bundle along tangent directions in the base manifold . Connections in this sense generalize, to arbitrary vector bundles, the concept of a linear connection on the tangent bundle of a smooth manifold, and are sometimes known as linear connections. Nonlinear connections are connections that are not necessarily linear in this sense.

Preliminaries 1.

We assume that all objects are smooth and all vector bundles are real throughout this paper. Let M be a manifold, $T(M)$ the tangent bundle. Let V and W be vector bundles over M. The fibre of V at $p \in M$ will be denoted by V_p and the dual bundle of V is denoted by V^*. The space of cross-sections of V will be denoted by $\Gamma(V)$. Let $H0M(V,W)$ be the vector bundle of which fibre $H0M(V,W)_p \; at \; p$ is the vector space $HOM(V_p, W_p)$ of linear maps from $V_p$to W_p. Especially $H0M(V,V)$ will be denoted by $END(V)$. Note that $H0M(V,W)$ can be naturally identified with the tensor product $V^* \otimes W$. The space of vector bundle homo-morphisms from V to W will be denoted by $HOM(V,W)$.

Definition 2.[ABE,1985**]**

Let V be a vector bundle over M. For $S \in \Gamma(E)$, we will denote the 1-jet of s by $j^1(s)$ and the 1-jet at p by $j_p^1(s)$. (General connections) : Let V be a vector bundle over M, Let$J^1(V)$ be the 1-jet bundle of V. A vector bundle homomorphism $\gamma \in HOM(V, J^1(V))$ is called a general connection on V. An endomorphism $P^r := \lambda \circ \gamma \in END(V)$ is called the principal endomorphism of γ. A linear operator$\nabla^r: \Gamma(V) \to \Gamma(T(M)^* \otimes V)$, de-defined by

$$\nabla^r s := \iota^{-1}(J^1(P^r s) - r(s)) \; for \; s \in \Gamma(V) ,$$

is called the covariant derivative of γ.

Definition 3. [Ali *et al*, 2012**]**

A connection on a vector bundle E is a map

$$D: \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$$

which satisfies the following conditions:

i. For any $s_1, s_2 \in \Gamma(E)$,
$$D_{s_1+s_2} = D_{s_1} + D_{s_2}$$
ii. For $s \in \Gamma(E)$ and any $\alpha \in C^\infty(M)$
$$D(\alpha s) = d\alpha \otimes s + \alpha D_s$$

Suppose X is a smooth tangent vector fields on M and $s \in \Gamma(E)$. let

$$D_{\mathrm{x}} s = < X, D_s >$$

Where$<,>$represents the pairing between TM and T^*M.

Then $D_{\mathrm{x}} s$ is a section of E, called the absolute differential quotient or the covariant derivative of the section s along X.

Theorem 4.[ABE,1985**]**

If ∇^r is the covariant derivative of a general connection γ with the principal endomorphism P^r, then

$$\nabla^r fs = (df) \otimes P^r s + f\nabla^r s \;\; for\, f \in C(M)\; s \in \Gamma(V)$$

Theorem 5.[Ali *et al.*, 2012**]**

Suppose D is a connection on a vector bundle E, and $p \in M$. Then there exists a local frame field$S$ in a coordinate neighborhood of p such that the corresponding connection matrix w is zero at p.

Theorem 6.[ABE,1985**]**

If $\nabla \in (V; P\,)$ for $P \in END(V)$, then there exists a unique

$\gamma \in HOM(V, J^1(V))$ such that $P^r = P \; and \; \nabla^r = \nabla$.

Theorem 7.[Ali *et al*, 2012**]**

Suppose $X, Y$are two arbitrary smooth tangent vector fields on the manifold MThen

$$R(X, Y) = D_{\mathrm{x}} D_Y - D_Y D_{\mathrm{x}} - D_{[\mathrm{x}, Y]}$$

Theorem 8.[Ali *et al.*, 2012**]**

The curvature matrix Ω satisfies the Bianchi identity

$$d\Omega = w \wedge \Omega - \Omega \wedge w\,.$$

Remark 9. [Ali *et al*., 2012**]**

If a section$s$ of a vector bundle Esatisfies thecondition $D_s = 0$, then s is called a parallel section.

Theorem 10.[Ali *et al*., 2012**]**

A connection always exists on a vector bundle.

Definition 11.[Ali *et al*., 2012**]**

Let M be a smooth n-dimensional manifold $O_M$be the set of smooth functions and $\Gamma(TM)$ be the vector space of smooth vector fields. An affine connection on $M$is a map (denoted by ∇)

$$\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$$
$$(X, Y) \mapsto \nabla_{\mathrm{x}} Y$$

Such that

i. $\nabla_{\mathrm{x}}(Y_1 + Y_2) = \nabla_{\mathrm{x}} Y_1 + \nabla_{\mathrm{x}} Y_2$
ii. $\nabla_{\mathrm{x}_1 + \mathrm{x}_2} Y = \nabla_{\mathrm{x}_1} Y + \nabla_{\mathrm{x}_2} Y$
iii. $\nabla_{\mathrm{x}}(f\, Y) = X(f) Y + f \nabla_{\mathrm{x}} Y$
iv. $\nabla_{f\mathrm{x}} Y = f \nabla_{\mathrm{x}} Y; \forall f \in O_M$ and $X, Y \in \Gamma(TM)$

The existence of parallel sections

Definition 1.[Ali *et al*., 2012**]**

The torsion of is the anti-symmetric tensor

$$T(X, Y) = \nabla_{\mathrm{x}} Y - \nabla_Y X - [X, Y]$$

where $[X, Y]$denotes the Lie brackets of the vector fields

$X$and Y; ∇ is called symmetric if $T = 0$.

Definition 2.[Atkins, 2011**]**

Let $\pi : W \rightarrow M$be a smooth vector bundle over a differentiable manifold M and let

$$\nabla : \mathcal{A}^0(W) \rightarrow \mathcal{A}^1(W)$$

Be a connection on W, where $\mathcal{A}^n(W)$ denotes the space of local sections $U \subseteq M \rightarrow W \otimes \Lambda^n M$. Weseekthesubsetof$W$

Generated by the local smooth parallel sections of W. Note that a local parallel section X must satisfy

$$R(\xi_1, \xi_2)(X) := \nabla_{\xi_1} \nabla_{\xi_2} X - \nabla_{\xi_2} \nabla_{\xi_1} X - \nabla_{[\xi_1, \xi_2]} X = 0$$

For all local vector fields ξ_1 andξ_2 defined with in the domain of the definition of X. Thus, we begin with the subset $V^{(0)}$of W consisting of all elements $w \in W$ that annihilate the Riemann curvature: $R(*,*)(w) = 0$. Then any local parallel section X of W is also a local section of $V^{(0)}$. By considering the kernel of these cond fundamental form of the smoothed-out part $W^{(0)}$of $V^{(0)}$, one obtains a subset $V^{(1)}$of $W^{(0)}$. Continuing to compute the kernel of the second fundamental form fore a chnew smoothed-out subset leads to a derived flag that will terminate at some subset, denoted $\widetilde{W}$. We claim that $\widetilde{W}$ is the sought for subset of W generated by the local parallel sections.

Theorem 3.[Atkins, 2011**]**

Let ∇ be a connection on the smooth vector bundle

$\pi : W \to M$.

i. If $X : U \subseteq M \to W$ is a local parallel section then the image of X lies in $\widetilde{W}$.
ii. Suppose that ∇ is regular at $x \in M$. Then for every$w \in W_x$ there exists a local parallel section$X : U \subseteq M \to \widetilde{W}$with$X(x) = w$.

Corollary 4. [Atkins, 2011**]**

Let ∇ be a regular connection on the smooth vector bundle $\pi : W \to M$. Then $(\widetilde{W}, \nabla)$ is a flat vector bundle over M.

Corollary 5.[Atkins, 2011**]**

Let ∇ be a connection on M, regular at $x \in M$. Then ∇ is locally metric at x if and only if $\widetilde{W}_x$ contains a positive-definite bilinear form.

Theorem 6. **[**Cantrijn, 2003**]**
Given a vector bundle $\nu : N \to M$, a vector bundle morphism $\rho : N \to TM$ such that $\nu = \tau_M \circ \rho$, and a fibre bundle $\pi : E \to M$. Then, there always exists a ρ-connection on π .

Conclusion

The theory of connections of vector bundles is important topics in differential geometry , and in the paper inception, we have mentioned about conduction on connection on a fibre bundle E over a manifold M. Some theories about Connections on the Parallel Vector Fields has presented in this paper.

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