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Preface to the 2^{nd} issue

As we live in exceptional circumstances in our country, Sudan, we seek peace and stability first. But each of us must look forward according to his specialization in turn, because life does not stop with such extraneous depressions. We expect that after peace prevails throughout our country, we will be in dire need of reconstruction and continuation of the renovation process at a steady pace to quickly overcome the negative effects of this war imposed on all of us.

While we work to find solutions and address the problems facing the process of construction and development in our country, it is necessary to continue the process of spreading valuable scientific knowledge by presenting researches that serves the issues of development, urbanization and future renaissance.

We are pleased to present to you the second issue of the Nile Journal of Science and Engineering. We hope that, your sober scientific contributions in all issues of the journal will be a real addition to the world of knowledge and scientific innovation, through which we can fulfill our commitment to the reader to provide all that is useful in the fields.

In this regard, I want to thank everyone who contributed and worked to put together this issue, and we hope that it will be a real addition to the information of the specialized reader.

Editorials

About the journal:

The Nile Journal for Science and Engineering (NJSE) is a bi-annual research journal aimed to publish original high quality research articles in the fields of Science and Engineering that are not published or not being considered for publication elsewhere. The journal is devoted to provide an appropriate forum for the dissemination of high-quality and high-impact original balanced credible academic writings in all disciplines of Science and Engineering. The work for publication (research papers, review articles, technical reports and short communications) will be accepted either in English or in Arabic.

Scopes

- ✓ Mathematics
- ✓ Physics
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The statements and opinions expressed in the articles herein are those of the author(s) and not necessarily of NJSE editorial board. All biological experimental works (such as genetic engineering) should be ethically acceptable and be in accordance with the local and international guidelines provided for both animal and human. Authors must guarantee that the manuscript parts were not being considered for publication elsewhere.

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Copyright

Instructions to authors

The work for publication will be accepted either in English (American or British flair isaccepted, but not a mixture of both) with an extra abstract in Arabic language or, in Arabic with an extra abstract in English. The manuscript should be written in clear, concise and grammatically correct language. Latin scientific names at all taxonomic levels are *italicized*. All scientific manuscripts will be reviewed by at least two reviewers. While submitting a revised manuscript, authors are requested to include a point to point response to reviewer's comments. The final decision relative to acceptance will be made by the editorial board. The corresponding author is considered the guarantor for the legitimacy and correctness of the manuscript as a whole.

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The following items should be handled in as printout hard copies and in electronically readable form or via e-mail:

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2. Manuscript:

The manuscripts submitted to the journal must conform to all style requirements stated by the Editorial Board.

Types of articles:

Full length original scientific papers: regular scientific papers, should report the results oforiginal research that have not being considered for publication elsewhere. A full research paper should have, in proper order, a Title, Abstract, Introduction, Materials and Methods, Results, Discussion, Conclusion and References.

Review papers**:** in which authors should review the up-to-date developments in relevant field of an active current interest. They have to be contemporary and comprehensive. Methods usedfor locating, selecting, extracting, and synthesizing data should be described. The abstract (250 words) should represent an accurate summary of the article.

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Regular scientific manuscript formatting

Manuscripts should be kept to a minimum length (the whole article length should not exceed 25 pages. The language of manuscript must be simple and explicit. Times New Roman fontstyle with 12 font size should be used. The whole manuscript should be typed double spaced (except tables), with all margins fixed at 2.5cm. Do not leave lines between paragraphs, butinstead, indent the first line of each paragraph with one tab space. All pages should be numbered consecutively in the bottom center. Each heading should appear on its own separate line, aligned to the left and bold. The use of footnotes should be avoided.

Title and author information: The title and author information should be centered across thetop of the first page. The title should give a concise designation of the paper topic, typed in 16font style. Capitalize the first letter of all main words (Latin names in *italics*). Abbreviationsshould not be used. This should be followed by the name(s) of the author(s) displayed as firstname, middle and last name (with no commas in between). Numbered and listed author'saffiliations will be provided immediately after the authors line flush left. The name of thecorresponding author should be underlined.

Abstract: The abstract, for reviews and research articles, as one paragraph, should be rational, informative and succinctly and clearly state the basic contents of the article with special focuson the scope, objectives, methods, and major findings. The abstract must be precisely andconsistently briefed, not exceeding 250 words. References and citations to other works shouldnot be included in the abstract.

Keywords: Key words (3-7 non generic words or phrases) should be provided directly belowthe abstract to express the article content. They should be written left aligned, arrangedalphabetically, separated with commas, capitalize only the first letters of proper nouns, and donot end the list with a full stop.

Abstract translation: An abstract translation is to be prepared and assembled below thekeywords, or at the beginning of the succeeding page. It should be precise and presents wordby- word translation.

The main text: The main text should be structured into paragraphs in the following order: **Introduction** including objectives, **Materials and Methods**, **Results**, **Discussion**, **Conclusion**, **Acknowledgement** (if any) and **References**. Results and discussion could becombined in one paragraph. Each heading should be flush with the left margin and have thefirst letter of all main words capitalized. All major headings should be separated from the textby 1 line space above.

Introduction: The introduction is a brief review of literature which should supply sufficientbackground information on the importance of the topic, the research area of the study and thehypotheses tested in the study. The specific aims of the project should be identified along withrationale for the specific experiments and other work performed. All sub- headings, if any, should be left justified, bold and title case. Objectives of the research should be clearly stated.

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Results –Discussion: They can be presented together (**Results and Discussion**) or in 2different sections (**Results** followed by **Discussion**). All results obtained should be simply and concisely presented in a logical order in figures, tables, or text. Tables and figures, whether integrated into the main text, or alternatively, printed on separate pages before the reference section, each should be sequentially numbered and titled (above for tables and below for figures). Do not present the same information in both a table and a figure. All printed tables and figures must be referred to in the text. Tabular data, were recommended, could be accompanied by either standard deviation values or standard errors of the means. The number of replicate determinations used for making such calculations must also be included. Sufficient statistical verification should be provided to identify differences in significance.

The results at hand should be concisely discussed in relation to hypotheses advanced inthe introduction section and interpreted to previously published works. It must not containextensive reiteration of the Introduction and Results sections.

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Acknowledgments: The source of any financial or technical assistant received for the workbeing published must be indicated in the Acknowledgments section.

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Please ensure that every reference cited in the text is also present in the reference list (and vice versa). Unpublished results and personal communications are not recommended in the reference list, but may be mentioned in the text. The NJAS in-text citation format is the familiar “author, date” format, e. g. Davis (1999) or (Davis, 1999). If the article has two authors, their last names are used, separated by the word “and”, e.g. Gates and Dobraszczyk (2004) or (Gates and Dobraszczyk, 2004). When there are more than two authors, only the first author’s last name should be mentioned, followed by “*et al*.”, e.g. Cao*et al.* (2015) or (Cao*et al.*, 2006).

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Examples:

Article in a Journal:

Sulieman, A.M.B; Warda, E.A.M. and Abdelgadir, S. (2015). Isolation and identification of yeasts from the different stages of *Hulu-mur* fermentation. Journal of Microbiology Research, 5(2): 71-76.

Book:

Allen, James P. (2008). Biophysical Chemistry. (1stEdition),Blackwell Publishing, Chichester, West Sussex, UK.

Book chapter:

Hardy, Jeffrey (2004). Green Chemsitry and Sustainability; Stevens, Christian V. and Verhe Roland, Eds.; Renewable Bioresources (Scope and Modification for Non-food Applications). John Willey and Sons Ltd. Chechister, Sussex, England. pp. 1-29.

Conference proceedings:

Trossero, M.A. (2000). Food and Biofuel Production. In: New Challenges Ahead. First World Conference on Biomass for Energy and Industry. Eds. Kyritsis, S.; Beenackers, A.A.; Helm, P.; Grassi, A. and Chiaramonti, D., pp.17-21. James and James Ltd., London.

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Figures should be properly sized and cropped so that no unnecessary white space is leftbordering the figure. Figures should be cited in the main text (appear where should be cited) in numerical order, or otherwise at the end of the manuscript. Figures should have titles set below the figure begins with the word “Figure or Fig.” followed by a number and a colon(:). Legends with sufficient details could be added. Multipanel figures (figures with parts labeled a, b, c, d, etc.) should be assembled into a composite as their final form. For an illustration taken from other publication, the source is to be cited.

Abbreviations and Units: SI units (metre, kilogram, etc.) should be used. Consistency mustbe maintained throughout the text in the use of abbreviations and units. Use standard abbreviations, that are accepted and recognized as common scientific terminology (hr, min, sec, etc.) instead of writing complete words. Define all non-standard abbreviations the first time they are used, then subsequently use the abbreviation.

Examples of some common abbreviations: Time: min, hr, sec; Length: km, m, cm, mm; Mass: kg, g, mg, μg; Concentration: g/cm3, g/L, mg/L, μg/L, ppm; Volume: cm3, L, mL, μL

TABLE OF CONTENTS

Nile Valley University Publications
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Evaluation of Superplasticizer Effects on the Properties of Workability and Water Absorption of Concrete Mixes

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Abstract

In order to satisfy the needs of engineers and contractors, this research aims to examine the effects of superplasticizers on the qualities of workability and water absorption of concrete mixes when combined with local materials. The methodology used to accomplish the research's goals focuses mostly on data gathering from many sources, a thorough evaluation of prior studies, creating concrete mixes, and conducting several laboratory trials utilizing various ratios of superplasticizer with low water concentrations. By creating 12 cubes for each sample, the ratios of superplasticizer applied were 0.0 (as a reference mix), 0.4, 0.8, 1.2, and 1.5 liters/50 kg cement for concrete ages of 3, 7, and 28 days. Graded natural coarse and fine with local ordinary Portland cement (OPC) were used for all concrete mixes conducted in this research. The experimental findings showed that the ratio of superplasticizers with a reduction of 15% (w/c) to 50 kg of cement (0.8 liters) significantly improved the workability of concrete. Additionally, there has been great workability with the ratios of 1.5L per 50 kg of cement with a 30% reduction in water/cement and 0.4L per 50 kg of cement with a 10% reduction in water/cement. However, the absorption values exhibited no variations with respect to the SP or w/c content ratios that were utilized. According to the results, superplasticizers increase the workability of the concrete mix while lowering the water-cement ratio to improve its characteristics.

Keywords: Superplasticizer; workability; absorption; concrete mixes; (W/C) Ratio.

تقييم أثر الملدنات الفائقة على خاصيتي التشغيلية وامتصاص الماء للخلطات الخرسانية

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المستخلص

من اجل استيفاء متطلبات المهندسين والمقاولين، هدف هذا البحث لاختبار أثر الملدنات الفائقة على جودة التشغيلية وامتصاص الماء بواسطة الخلطات الخرسانية عندما تدمج بالمواد المحلية. المنهجية التي استخدمت للوصول لهدف البحث المنشود تركزت على الآتي: البيانات التي تم الحصول عليها من عدة مصادر، عبر تقييم الدراسات السابقة، عمل خلطات خرسانية، عمل عدة تجارب معملية بناءً على أخذ عدة نسب من الملدنات الفائقة وهي 0.0 (كنسبة مرجعية)، 0.4، 0.8، 1.2 و1.5 لتر لكل 50 كيلوجرام من الأسمنت ولخرسانة بعمر 3يوم، 7 يوم و28 يوم. في هذا البحث تم استخدام الركام المتدرج الخشن والناعم مع اسمنت بورتلاندي عادي. النتائج المعملية التي تم الحصول عليها أظهرت أن استخدام الملدنات الفائقة بنسبة تخفيض 15% من نسبة الأسمنت للماء لكل 50 كيلوجرام أسمنت (0.8 لتر)، تؤدي لتحسين تشغيلية الخرسانة بشكل ملحوظ. بالإضافة إلى أنه يوجد تشغيلية جيدة مع نسبة 1.5 لتر لكل 50 كيلوجرام من الأسمنت مع تخفيض بنسبة 30% من نسبة الماء للأسمنت و 0.4 لتر لكل 50 كيلوجرام من الأسمنت مع نسبة تخفيض 10% من نسبة الماء للأسمنت. مع ذلك فإن قيم الامتصاص لم تظهر أي تباين فيما يتعلق بالملدنات الفائقة أو نسبة الماء للأسمنت التي تم استخدامها. وفقاً لهذه النتائج يمكن القول إن الملدنات الفائقة تزيد من تشغيلية الخلطات الخرسانية بينما تقلل من نسبة الماء للأسمنت مما يحسن خصائصها.

كلمات مفتاحية:الملدنات الفائقة، التشغيلية، الإمتصاص، الخلطات الخرسانية، نسبة الماء للأسمنت.

Introduction

Admixtures are defined as additional materials for concrete constituents, i.e., cement, aggregates and mixing water, these constituents are added to the batch immediately before or during mixing (ASTM 125, 1999).

Superplasticizers are used in concrete to improve some properties of fresh and hardened concrete. These properties include, but not limited to, workability improvement, curing time increasing or decreasing, and increasing concrete strength. Sometimes admixtures are used to change the cement color.

A study on the dosage effect of superplasticizer on self-compacting concrete was reported by Benaicha *et al*. (2019) one of several researchers who examined the impact of superplasticizer on the properties of concrete mixes (the correlation between rheology and strength). Slump flow, V-funnel, L-box, yield stress, and plastic viscosity were employed to measure rheology. in addition to compressive strength, mechanical tests. The study's findings indicate that the tests had a strong correlation.

To assess the performance of self-compacting concrete using 10% fly ash substitution for cement in addition to varied doses of superplasticizer, Dumne(2014) investigated the effect of superplasticizer on the fresh and hardened properties of the material. The study found that the combination of fly ash and superplasticizer improved workability and raised compressive strength.

In an effort to determine the ideal superplasticizer dosages to improve concrete's strength and workability, Alsadey(2015) made an effort to research the impact of superplasticizer on the fresh and hardened qualities of concrete. Improved workability and compressive strength are the results of the research.

Superplasticizer impact in mineral dispersal systems based on quarry dust was assessed by Smirnova (2018). The compatibility issue was examined when this admixture was used with microfillers made from quarry dust from rock crushing.

Additionally, studies on the impact of superplasticizers alone or in combination with other admixtures on concrete mixes were undertaken by Saeed Ahmad *and* Elahi, Ayub (2005); Mbadike (2011); E., Alsadey (2012); Gayathri (2014); Fadhil Nuruddin *et al*. (2011).

Materials Used and Method of Testing

Cement

Ordinary Portland Cement (OPC), which complies with British Standards (BS 1996) is utilized in this study. It was tested for its physical characteristics, including as its standard consistency, start and final setting times, and compressive strength. It was discovered that 29% of the cement's weight came from the water absorbed via the routine consistency test. The beginning and final setting times of cement were experimentally determined using this amount of water. Table 1 displays the findings of the physical characteristics and compressive strength of cement mortar cubes aged 2 and 28 days. These studies' findings demonstrate that the cement utilized in this study complies with British Standards (BS 1296).

Aggregates

For this study, concrete mixtures were made using natural aggregates (both coarse and fine). The grade of aggregates to be utilized in concrete mixtures was investigated using sieve analysis tests. According to British Standards BS (1992); 882:1992 and (BS, 1992) 812-103, these tests were conducted. Tables (2) and (3), respectively, give the results for coarse and fine aggregate. The grading of coarse aggregate met BS 812-103.1 (882:1992), as shown in Table (2) and Fig. 1, whereas the grading of fine aggregate met BS 882:1992 and BS 812-103, as shown in Table (3) and Fig. 2.

Admixture used

The superplasticizer caplast super-special, which complies with ASTM C494 Type F (ASTM, C494) and BS 5075 Part III (BS, 1985) was utilized in this study as a water reducer. Utilizing this kind of admixture aims to decrease mixing water, provide excellent workability, and greatly boost compressive strength.

Results and Discussion

In this study, intensive laboratory tests were carried out to investigate the effects of the superplasticizer on the properties of fresh concrete (workability) and the water absorption of the concrete mix. Preliminary tests for local ordinary Portland cement and aggregate used in the research have been carried out. The ratios of superplasticizer added were 0.0 (as a reference mix), 0.4, 0.8, 1.2, and 1.5 liters/50 kg cement for concrete ages of 3, 7, and 28 days by preparing 12 cubes for each sample. The fresh concrete mixes were cast in standard test molds of 150 mm cubes, whereas a standard slump cone of 300 mm high and 150 mm in diameter was used for measuring concrete slumps. The results of these experiments are shown in the following tables and figures:

Preliminary Tests Results of Cement

The results of the cement paste's physical characteristics and compressive strength are displayed in Table 1.

Table (1): Results of Preliminary Cement Tests

Test	Results	Requirements of BS 12 1996
Consistency	29.0%	26% -32%
Setting Time		
a. Initial setting time (min)	60	Not less than 60 min
b. Final setting time (hr)	3.17	Not more than 10 hrs.
Compressive Strength		
a. 2 days		
Sample-1	17.6 N/mm^2	Equal or Greater than 10 N/mm^2
Sample-2	17.2 N/mm^2	
Sample-3	17.32 N/mm^2	
b. 28 days		
Sample-1	45.6 N/mm^2	Equal or Greater than 42.5 N/mm^2
Sample-2	44.1 N/mm^2	
Sample-3	46.2 N/mm^2	

Results of Aggregate Tests

Natural fine aggregate was created in accordance with BS 5075-1:1982, while natural uncrushed coarse aggregate was created in accordance with British Standards BS (882:1992). The outcomes of the sieve analysis test for coarse aggregate conducted in accordance with BS 812-103 (BS 812-103.1: 1992) are displayed in Table 2. The greatest aggregate, which can be determined from Table (2), is 20 mm, and the sample appears to be well-graded, as Fig. 1's vivid illustration makes evident. The outcomes of the fine aggregate sieve analysis are displayed in Table 3 and Fig. 2.

Table (2): Results of Sieve Analysis of Coarse-Aggregate

B.S Sieve (mm)	Retained					%Age Passing	BS 812-103
	Sample (1)		Sample (2)		Average		
	Wt. (g)	(%)	Wt. (g)	(%)	(%)		
25	0	0	0	0	0	100	100
19.5	0.129	6.45	0.131	6.55	6.5	93.5	85 to 100
12.5	1.113	55.6	1.265	63.25	59.425	34.075	0 to 70
9.5	0.567	28.35	0.450	22.5	25.425	8.65	0 to 25
4.75	0.185	9.25	0.155	7.75	8.5	0.15	0 to 5
pan	**0.006**	**0.3**	**0.0**	**0**	**0.15**	**0**	**-**

Absorption Ratio % = (1002-1000)/1000 = 0.2%

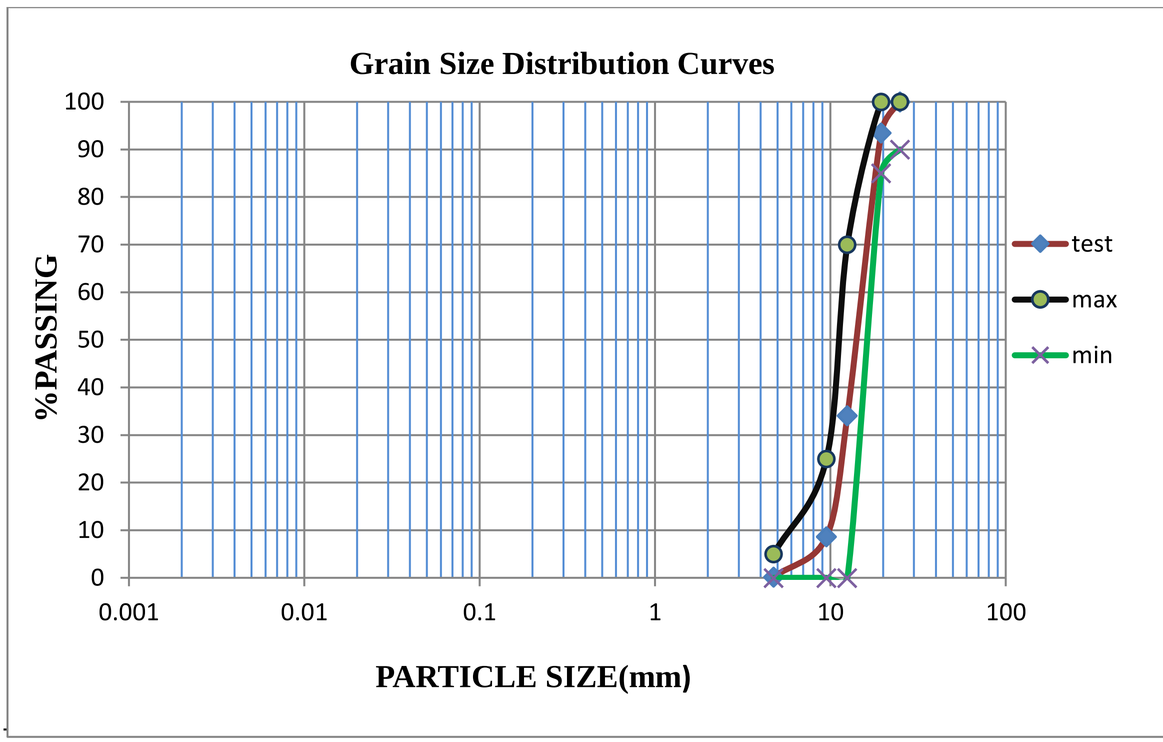

Figure (1): Grain Size of Coarse-Aggregate Test

From Figure (1), the distribution curves resulted in well-graded coarse aggregate.

Table (3): Results of Sieve Analysis of Fine-Aggregate Test

B.S Sieve (mm)	Retained					%Age Passing	BS 812-103
	Sample (1)		Sample (2)		Average		
	Wt. (g)	(%)	Wt. (g)	(%)	(%)		
4.75	0.016	1.6	0.019	1.9	1.75	98.25	89 to 100
2.36	.029	2.9	0.031	3.1	3	95.25	60 to 100
1.18	.0052	5.2	0.052	5.2	5.2	90.05	30 to 100
0.600	.0158	15.8	0.161	16.1	15.95	74.1	15 to 100
0.300	.0626	62.6	0.656	65.6	64.1	10	5 to 70
0.150	.0096	9.6	0.062	6.2	7.9	2.1	0 to 15a
0.075	0.023	2.3	0.019	1.9	2.1	0.4	-
pan		00	00	00	00	**00**	**00**

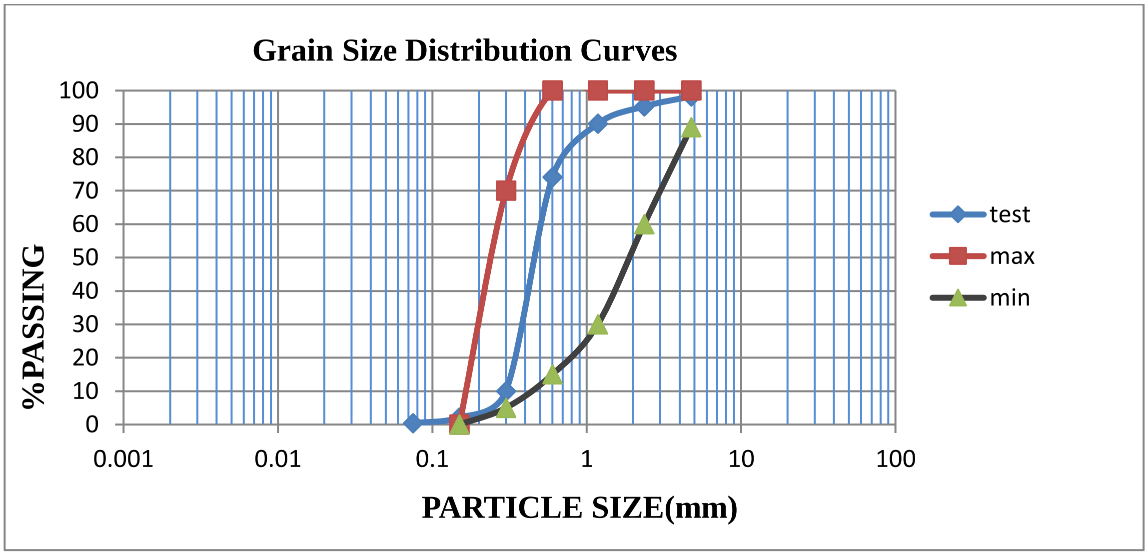

Figure (2): Grain Size of Fine-Aggregate Test.

Table (4): Fine Aggregate Silt Content

Sample	Sample 1	Sample 2
Total weight (g)	1000	1000
Weight after washing (g)	992	991
Silt and clay (%)	0.8	0.9

Results of Workability (Slump) for Fresh Concrete Tests

The equipment for the slump test is indeed very simple. It consists of a tamping rod and a truncated cone, 300mm in height, 100mm in diameter at the top, and 200mm in diameter at the bottom. To conduct a slump test, first moisten the slump test mold and place it on a flat, nonabsorbent, moist, and rigid surface. Then hold it firmly to the ground by foot supports.

Next, fill 1/3 of the mold with the fresh concrete and rod it 25 times uniformly over the cross section. Likewise fill 2/3 of the mold and rod the layer 25 times, then fill the mold completely and rod it 25 times. If the concrete settles below the top of the mold, add more. Strike off any excessive concrete. Remove the mold immediately in one move. Measure and record the slump as the vertical distance from the top of the mold to average concrete level.

Table (5): Results of Slump Improved by Superplasticizer (SP) Ratios and Water Reduction

Superplasticizer (SP) (Litre/50-kg Cement)	**0.0**	**0.4**	**0.8**	**1.2**	**1.5**
Water Reduction (%)	0	10	15	20	30
Slump (mm)	40	90	180	190	185

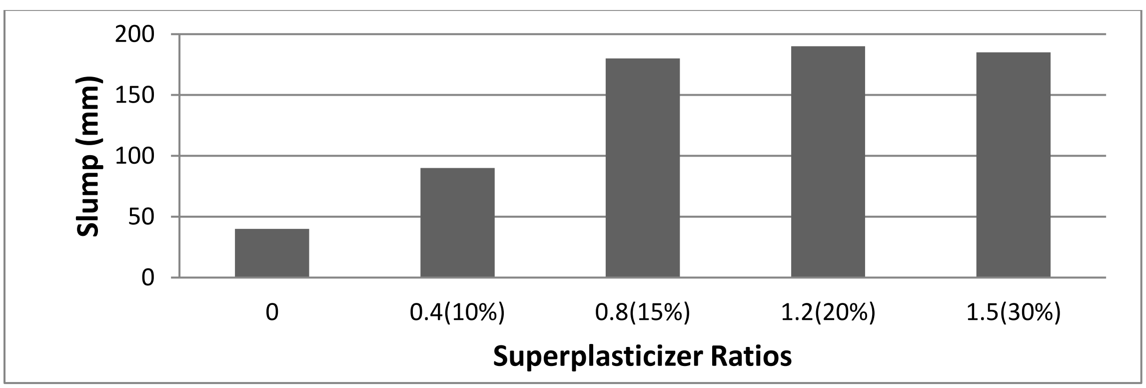

Figure (3): Relationship between Results of Slumps and Superplasticizer Ratios and Water Reduction

Table (5): Results of Slumps Improved by Superplasticizer Ratios with Water and Cement Reduction

Superplasticizer (Litre/50-kg Cement)	0.0	0.4	0.8
Water Reduction (%)	0	10	15
Cement Reduction (%)	0	10	15
Slump (mm)	40	50	85

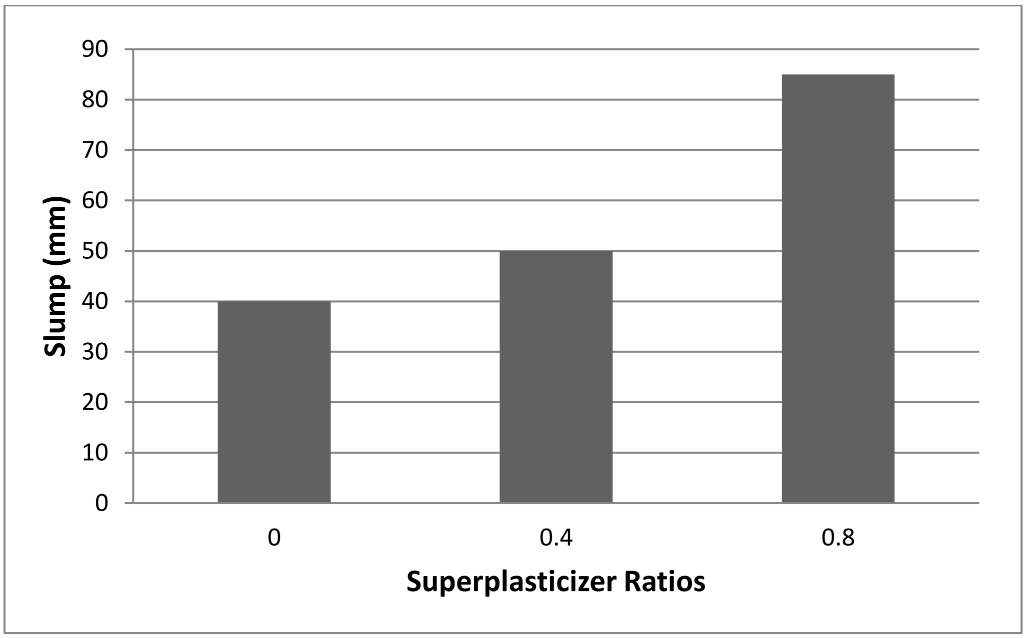

Figure (4): Relationship between Results of Slumps and Superplasticizer Ratios and Water & Cement Reduction.

Results of Concrete Water Absorption Tests

Drying the produced specimens until the mass was constant (WO). The specimens were then submerged for 28 days in clean water. The specimens were removed when the necessary immersion time had passed, the surfaces were swiftly cleaned with water, and they were then immediately weighed (W1), as shown in Tables 7–14. The following formula can be used to determine the rate of water absorption:

Water Absorption (%) = {(W1 – WO)/WO} *100.

Table (7): Results of Absorption after 28-Days: Control Mix

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	8.002	8.347	0.345	**3.42%**
2	8.021	8.251	0.23	
3	8.088	8.337	0.249	
Mean Weight	**8.037**	**8.312**	**0.275**	

Table (8): Result of Absorption after 28-Days: Superplasticizer Ratio = 0.4L/50-kg of Cement and 10% Water Reduction.

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	7.954	8.320	0.366	**3.33%**
2	8.283	8.492	0.209	
3	8.167	8.405	0.238	
Mean Weight	**8.135**	**8.406**	**0.271**	

Table (9): Result of Absorption after 28-Days: Superplasticizer (SP) Ratio = 0.8L/50-kg of Cement and 15% Water Reduction

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	8.032	8.333	0.301	**3.62%**
2	8.039	8.318	0.279	
3	8.203	8.501	0.298	
Mean Weight	**8.091**	**8.384**	**0.293**	

Table (10): Result of Absorption after 28-Days: Superplasticizer (SP) Ratio = 1.2L/50-kg of Cement and 20% Water Reduction.

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	8.326	8.624	0.298	**3.78%**
2	8.347	8.679	0.332	
3	8.365	8.683	0.318	

Mean Weight	**8.346**	**8.662**	**0.316**	

Table (11): Results of Absorption after 28-Days: Superplasticizer (SP) Ratio = 1.5L/50-kg of Cement and 30% Water Reduction

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	8.187	8.374	0.187	**2.21%**
2	8.352	8.548	0.196	
3	8.220	8.401	0.181	
Mean Weight	**8.253**	**8.441**	**0.188**	

Table (12): Result of Absorption after 28-Days: Superplasticizer (SP) Ratio = 0.4L/50-kg of Cement and 10% Water Reduction and 10% Cement Reduction

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	8.489	8.782	0.293	**3.95%**
2	8.220	8.545	0.325	
3	7.951	8.308	0.357	
Mean Weight	**8.220**	**8.545**	**0.325**	

Table (13): Result of Absorption after 28-Days: Superplasticizer (SP) Ratio = 0.8L/50-kg of Cement and 15% Water Reduction and 15% Cement Reduction

Cube No.	Initial Weight	Final Weight	Differences	Absorption
1	8.529	8.653	0.124	**1.44%**
2	8.391	8.518	0.127	
3	8.667	8.787	0.12	
Mean Weight	**8.529**	**8.653**	**0.124**	

Table (14): Effect of Superplasticizer (SP) Ratios on Absorption of Concrete Mixes for 28-Days of Age

No.	Dosage	Water Reduction (%)	Absorption (%)	Difference (Relative to Control Mix)	% Age Difference
1	0.0	0.0	3.42	3.42-3.42 =0	0
2	0.4	10.0	3.33	3.33-3.42 =-0.09	3 (decrease)
3	0.8	15.0	3.62	3.62-3.42=0.20	6 (increase)
4	1.2	20.0	3.78	3.78-3.42 =0.36	11 (increase)
5	1.5	30.0	2.21	2.21-3.42 =-1.21	35 (decrease)

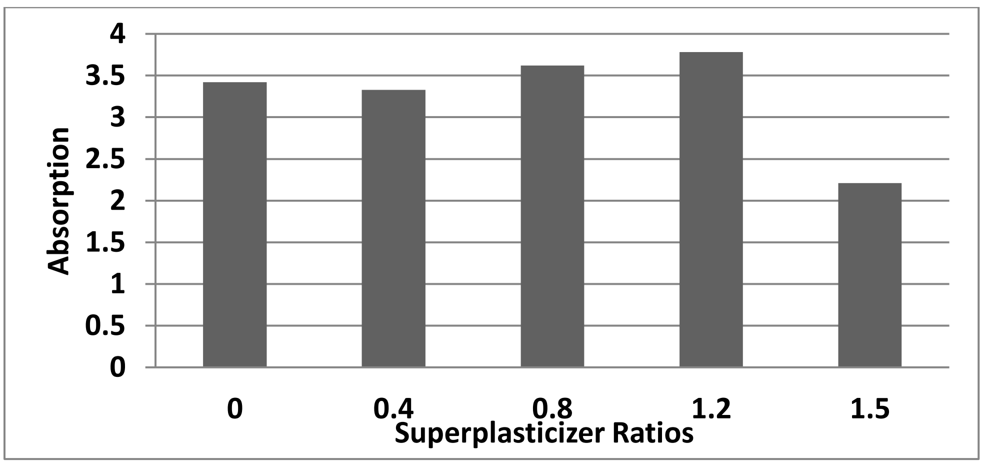

Figure (5): Relationship between Superplasticizer (SP) Ratios and Absorption of Concrete Mixes for 28-Days of Age

Table (15): Effect of (0.4 and 0.8) L/50-kg of Cement Superplasticizer, (10 and 15%) Water Reduction and (10 and 15%) Cement Reduction on Absorption of Concrete Mixes for 28-Days of Age.

No.	Dosage	Water Reduction (%)	Cement Reduction (%)	Absorption	Difference(Relative to control mix)	% Age Difference
1	0.0	0.0	0.0	3.42	3.42-3.42 = 0	0
2	0.4	10.0	10.0	3.95	3.95-3.42 = 0.53	15 (increase)
3	0.8	15.0	15.0	1.44	1.44-3.42 = -1.98	58 (decrease)

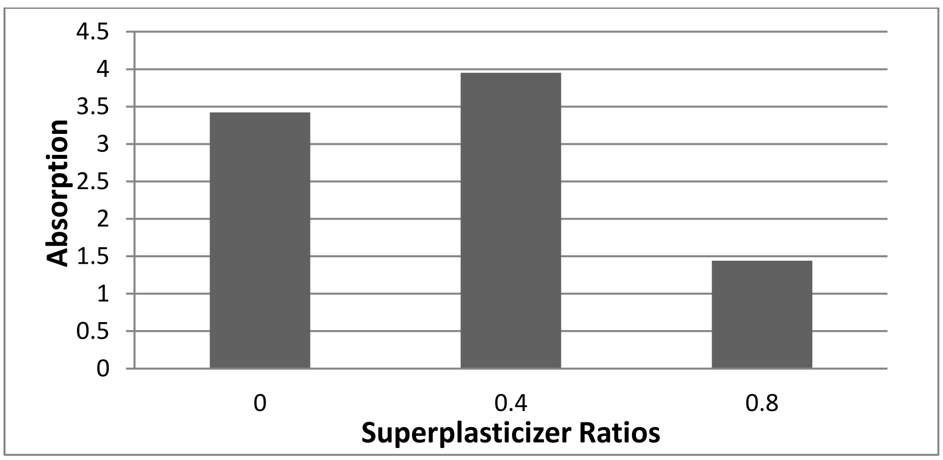

Figure (6): Absorption Concrete Mixes after 28-Days: Superplasticizer (0.4 and 0.8)L/50-kg of Cement, (10% and 15%) Water Reduction and (10% and 15%) Cement Reduction

Discussion of the Results:

Preliminary results of concrete constituents:

To ensure that the primary components of concrete (cement and aggregates) are sufficient and adhere to BS (British Standards) code requirements, laboratory tests were carried out. According to Table 1, which summarizes the findings of these preliminary tests, the cement utilized in this study conforms with BS 1296. Tables (2) and (3), respectively, give the results for coarse and fine aggregate. The grading of coarse aggregate met BS 812-103.1 (882:1992), as shown in Table (2) and Fig. 1, whereas the grading of fine aggregate met BS 812-103 and 882:1992, as shown in Table (3) and Fig. 2.

Workability

When admixtures were added to a common reference mix with a w/c ratio of 0.48, the slump test was utilized as a gauge of consistency. Slump values for standard reference mixes are 40 mm, while they are 90 mm, 180 mm, 190 mm, and 185 mm for mixes containing 0.4, 0.8, 1.2, and 1.5 L of superplasticizer per 50 kg of cement and reduced water by 10%, 15%, 20%, and 30%, respectively. A reduction in the slump in the proportion of 1.5 as a result of a significant reduction in the proportion of w\c compared with other ratios presented in Table (5) and Fig (3).

As indicated in Table 6 and Figure 4, slumps of mixes comprising (0.4 and 0.8 liters of superplasticizer per 50 kg of cement) reduced cement by 15% and water by 10%, respectively. While the workability of mixes combining admixtures and decreased cement was lower than that of blends containing only admixtures, it was still significantly greater than that of the common reference mix.

Water Absorption

Table (14) demonstrates that the values of absorption vary regardless of superplasticizer ratios or water reduction rates. Table (15) demonstrates that as superplasticizer ratios increased (0.4% and 0.8%) and cement and water contents reduced, absorption values decreased.

Conclusion

This investigation was done to find out how superplasticizer (SP) affected the characteristics of freshly-poured and hardened concrete.

Compressive strength and workability (slump) were the qualities that were examined. The study's findings lead to the following conclusions:

- In all ratios of this admixture added to concrete mixtures, superplasticizer had a substantial impact on the characteristics of fresh concrete.
- The slumps of mixes including ratios of SP with reduced water content were observed to range from 90 mm to 190 mm, which is significantly greater than the slump of the standard reference mix (40 mm) and almost twice as high as that of mixes containing SP with reduced water and cement (slumps reached up to 85 mm). However, very high doses of SP have a tendency to make concrete less cohesive.
- With regard to the applied ratios of SP or w/c contents, it was found that the absorption values did not exhibit any steady state alterations.

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دراسة نظام السلامه والصحة المهنية بمصانع الاسمنت بولاية نهر النيل

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المستخلص :

تُعَدُّ السلامةُ والصحةُ المهنية في مجال صيانة مصانع الاسمنت من العناصر المهمة لارتباطها بالمخاطر، ومعظم مصانع الأسمنت في السُّودان لم تبدِ اهتماماً وأولوية بنظام السلامة ولذلك اتجهت الدراسة بالبحث في أنظمة السلامة في مصانع نهر النيل، فتناولتْ آراء العاملين حول أنظمة السلامة بمصانع أسمنت ولاية نهر النيل، وقد هدفتْ من ذلك استطلاع آراء العاملين حول نظام السلامة في خط الإنتاج لمصانع ولاية نهر النيل، ومناقشتها وتحليلها والوصول إلى نتائج تسهم في تحسين نظام السلامة والصحة بمصانع أسمنت الولاية. وتناولت الدراسة مقدمة عن السلامة شملت عناصر الإنتاج وأهمية السلامة، وأهداف البحث، ومشكلة البحث، كما اعتمدتْ على المنهج الوصفي، واستبانات وُزِّعَتْ على جهات مختلفة. وقد توصلت الدراسة إلى العديد من النتائج منها:وجود مخاطر في مصانع الأسمنت بولاية نهر النيل منها: مخاطر السيور الناقلة- مخاطر حركة الآليات- مخاطر الكهرباء- مخاطر الحريق- مخاطر مرافق الإنتاج.

كلمات مفتاحية: السلامة، الصحة المهنية، المخاطر ، تقييم المخاطر ، النيبوش.

Study of the Occupational Safety and Health System for Cement Factories in the River Nile State

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Abstract

Occupational safety and health in the field of maintenance of cement factories is one of the important elements due to its association with risks, and most of the cement factories in Sudan did not show interest and priority in the safety system. The aim of this paper is survey workers' opinions about the safety system in the production line of the Nile River State factories, discuss and analyze them, and reach results that contribute to improving the safety and health system in the state's cement factories. The study dealt with an introduction to safety, which included the elements of production, the importance of safety, the research objectives, and the research problem. It also relied on the descriptive approach, and questionnaires distributed to different parties. The study reached many results, including: There are risks in the cement factories in the Nile River state, including: conveyor belt risks - machinery movement risks - electricity risks - fire risks - production facilities risks.

Keywords: Safety, occupational health , hazards, risk assessment, Nebosh

مقدمة

تُعَدُّ السلامة والصحة المهنية والاهتمام بها في أي مؤسسة مظهراً من مظاهر التطور الإداري والتخطيط الاقتصادي الناجح، كما تُعَدُّ انعكاسًا للوعي العام، وهي بالمفهوم الحديث الشامل تعني المحافظة على عناصر الإنتاج الرئيسة وهي: الإنسان (القوى العاملة داخل المؤسسة وخارجها) ، المواد الخام والمواد المنتجة،المحافظة على الآلات والماكينات وأدوات الإنتاج.

تتلخّص أهداف السلامة والصحة المهنية في حماية القوى البشرية والمادية المتمثلة في المنتجين والمهندسين والماكينات والآلات والمعدات من أي ضرر أو تلف يلحق بهم أو لتخفيض في مستوى جودة الإنتاج من جرّاء وقوع حوادث وإصابات العمل، وإزالة مسببات الخطر، وتأمين بيئة عمل آمنة خالية من المخاطر والأمراض المهنية (حلمي و العفشوك،2004).

أهداف البحث:

- دراسة نظام السلامة في صيانة خط الإنتاج لمصانع ولاية نهر النيل.
- دراسة المخاطر في المنطقة المحددة.
- تقييم تلك المخاطر(تناولت الورقة مصنعين).

مشكلة البحث

عدم وجود الاهتمام الكافي بالسلامة والصحة المهنية في المصانع والذي ينعكس سلبا على صحة العامل وعلى المنشأة من حيث الممتلكات، ووجود المخاطر تشكل خطرًا كبيرًا على العاملين.

منهج البحث

اعتمدت الدراسة على المنهج الوصفي، وإجراء مقابلة مع بعض الأطراف (مدير المصنع، مدير السلامة، بعض مهندسي خط الإنتاج)، كما اعتمدت أيضًا على استبانات وُزِّعت على جهات مختلفة. تم التحليل بواسطة برنامج (SPSS) في تحليل الاستبانات، كما اعتمدت أيضا على نظام النيبوش في تقييم المخاطر، وشملت هذه العينة (50) فردا بمختلف الوظائف (عامل، ومهندس، ومدير) عشوائيًا من منطقة الدراسة المستهدفة (الفرن- طاحونة الحجر- طاحونة الأسمنت)، لكل من المصانع المستهدفة ورمزلها بD/C/B/A.

دراسات سابقة

دراسة (بوسعيد سوهيلة،2015) بعنوان دور إدارة السلامة والصحة المهنية في تحسين أداء العاملين بالمؤسسات الصغيرة والمتوسطة الصناعية: دراسة حالة مؤسسة تحويل البلاستيك.

وهدفت هذه الدراسة إلى التعرّف على مدى مساهمة إدارة السلامة و الصحة المهنية في تحسين أداء العاملين في مؤسسة تحويل البلاستيك، وقد تم ذلك من خلال دراسة مسحية باستخدام الاستبانة، واستخدم في نتائج تحليل الدراسة الحزمة الإحصائية للعلوم الاجتماعية SPSS، وتوصلت الدراسة إلى ضرورة الاهتمام بالمورد البشري؛ وذلك عن طريق توفير ظروف آمنة داخل المؤسسات الصناعية لما له دور في أدائهم، وأن العلاقة الجيدة بين مدير إدارة السلامة والعاملين تساهم في تحسين أدائهم، وضرورة إعداد الدورات التدريبية في مجال السلامة لما له دور كبير في تحسين الأداء.

دراسة (مشعلي بلال،2010) دور برامج السلامة المهنية في تحسين أداء العاملين بالمؤسسات الصغيرة والمتوسطة الجزائرية: دراسة حالة مؤسسة StapapAlifلتحويل الورق إلى البلاستيك.

هدفت هذه الدراسة إلى التعرف على مدى مساهمة برامج السلامة في تحسين أداء العاملين في مؤسسة stapapalif لتحويل الورق والبلاستيك وقد تم ذلك من خلال دراسة عينة عشوائية باستخدام الاستبانة واستخدم في تحليل نتائج الدراسة الحزمة الإحصائية للعلوم الاجتماعية SPSS وتوصلت إلى نسبة تعرض العمال للحوادث المهنية كبيرة، كما أنّ ظروف العمل السيئة تمثل أهم الأسباب المؤدية إلى وقوع حوادث العمل إلاّ أن

معظم العمال يؤكّدون أنه توجد إمكانية لتفادي جميع الأسباب المؤدية لحوادث العمل وآخر ما توصّل إليه أنّ المؤسسة مقصرة نوعًا ما في تطبيق برامج السلامة.

دراسة (أُميمة صقر المغني،2006) بعنوان: واقع إجراءات الأمن والسلامة المهنية المستخدمة في منشآت قطاع الصناعات التحويلة في قطاع غزة.

هدفت هذه الدراسة إلى التعرّف على الواقع الذي تعيشه منشآت قطاع الصناعات التحويلة في قطاع غزة من حيث التزامها لتطبيق القوانين الخاصة بالسلامة والصحة المهنية، وقد تم ذلك من خلال دراسة عشوائية باستخدام الاستبانة، واستخدم في تحليل نتائج الدراسة الحزمة الاحصائية للعلوم الاجتماعية SPSS، وتوصلت إلى أنّ هناك علاقة ارتباطية ذات دلالة إحصائية بين فعالية إجراءات السلامة والصحة المهنية وبين الالتزام بتطبيق وتوفير الأنظمة واللوائح والقوانين الخاصة بالسلامة والصحة المهنية علي صعيد المؤسسات الرقابية والصناعية.

دراسة (علي موسي حنان، 2007) بعنوان الصحة والسلامة المهنية وأثرها على الكفاءة الإنتاجية في المؤسسة الصناعية، دراسة حالة مؤسسة هنكل- الجزائر مركب شغلوم العيد.

هدفت هذه الدراسة إلى التعرف على أثر الصحة والسلامة المهنية على الكفاءة الإنتاجية في المؤسسة الصناعية، وتوصلت إلى أنّ الصحة المهنية مجال يهتم بحماية العنصر البشري من أخطار حوادث العمل والأمراض المهنية وأن دوافع الاهتمام بهذا المجال تتمثل في دوافع إنسانية واقتصادية واخيرا توثر الحوادث والأمراض المهنية سلبا على الكفاءة الإنتاجية في المؤسسة الصناعية بسبب التكاليف المباشرة وغير المباشرة.

دراسة(هدار بختة،2012) بعنوان: دور معايير السلامة والصحة المهنية في تحسين أداء العاملين في المؤسسات الصغيرة والمتوسطة: دراسة حالة مؤسسة ليند غازالجزائر- وحدة ورقلة.

هدفت هذه الدراسة إلى الإلمام بكافة جوانب السلامة وتقييم الموارد البشرية في المؤسسات الصغيرة والمتوسطة، وتوضيح العلاقة بين زيادة أداء العاملين ووجود نظام السلامة والصحة المهنية في المؤسسات الصغيرة والمتوسطة، وتوصلت إلىأن إدارة الصحة والسلامة تعمل على المحافظة العاملين، وتوفر بيئة مناسبة للعمل من أجل رفع إنتاجية العاملين، هنالك مجوعة من المعايير التي يعتمد عليها التصنيف في المؤسسات وتحديد نوعها، وتوصّل أخيراً إلى أنّ هنالك تطبيقًا لمعايير السلامة في بعض المؤسسات ولكن يوجد إهمال من العاملين.

دراسة(محمد عمر عبد الواحد،2016) بعنوان: تطبيقات إجراءات السلامة والصحة المهنية في المنشآت التعليمة: دراسة حالة مبنى كلية علوم الحاسوب جامعة الزعيم الأزهري- السودان الخرطوم.

هدفت هذه الدراسة الي معرفة مدى تطبيق الإجراءات الخاصة بالسلامة والصحة المهنية في المنشآت التعليمة ومعرفة مدي الإلتزام بتطبيق هذه الإجراءاتمن قبل الجهات ذات الإختصاص في كلية علوم الحاسوب بجامعة الزعم الازهري, كما هدفت الى دراسة المخاطر المختلفة المحيطة بالمنشآت التعليمة ومعرفة القوانين والتشريعات الكفيلة بمنع وقوع الحوادث, استخدم الباحث في هذه الدراسة المنهج الوصفي وتوصلت الى وجود ضعف واضح في متطلبات بنود الإشتراطات المتعلقة بمتطلبات الأمن والسلامة في المباني التعليمة.

دراسة(هيثم الطاهر محمد،2015) بعنوان: ثقافة السلامة والصحة المهنية في العمل الهندسي - السودان

هدفت هذه الدراسة الي معرفة اهمية ثقافة السلامة والصحة المهنية والاهتمام بصحة العمال في العمل الهندسي والآثار المترتبة علي عدم تطبيق قوانين السلامة والصحة المهنية. استخدمت الدراسة منهجية الفرضيات التي قيست مدى المعرفة بثقافة السلامة والصحة المهنية والاهتمام بصحة العمال والإلتزام بتنفيذ قوانينها والعمل بها. توصلت الدراسة الى اثبات الفرضية التي تنص علي أن هناك معرفة بثقافة السلامة والصحة المهنية والاهتمام بصحة العمال والإلتزام بتنفيذ قوانينها والعمل بها بدرجة عالية.

دراسة المصانع

المصنع A:

نتائج تحليل المصنع A

اتضح من خلال تحليل بيانات المصنع (A) من خلال الاستبانة المطروحة على عينة البحث ما يأتي:

*** فيما يختص بطبيعة العمل ومواءمته لمعايير السلامةأظهرت النتائج أنّ**

- 76% يوافقون على أن مستوى الإضاءة مناسب جداً لطبيعة العمل، و8% يوافقون على أن مستوى الإضاءة غير مناسب لطبيعة العمل، و16% محايدين.
- 76% يوافقون على أن مستوى التهوية مناسب جداً لطبيعية العمل، و14% يوافقون على أن مستوي التهوية غير مناسب، و10%محايداً.
- 36% يوافقون على أن الضجيج لا يؤثر على سلامتهم، و28% أن الضجيج يؤثر على سلامتهم و36% محايداً.
- 54% يوافقون على أن مستوى الاهتزاز مناسب مع طبيعة العمل، و24% يوافقون على أن مستوي الاهتزاز غير مناسب مع طبيعة العمل، و22% منهم محايداً.
- 54% يوافقون على أن درجة الحرارة مناسبة مع طبيعة العمل، و24% يوافقون على أن مستوى درجة الحرارة غير مناسبة مع طبيعة العمل، و22% محايدًا.
- 76% يوافقون على أن مكان العمل آمن من مخاطر الكهرباء، و8% يوافقون على أن مكان العمل غير آمن من مخاطر الكهرباء، و18% محايداً.
- 72% يوافقون على أن عند وقوع حادث يمكنهم التصرف معه، و4% يوافقون على أنهم لايمكنهم التصرف عند وقوع حادث، و24% محايدًا.
- 72% يوافقون على إمكانية الوصول لأجهزة الإنذار والإطفاء في زمن قياسي عند وقوع حريق، و18% يوافقون على عدم وصولهم لأجهزة الإنذار والإطفاء في زمن قياسي، و20% محايداً.
- 76% يوافقون على أن يمكنهم التصرفوالوصول السريع إلى مخارج الطوارئ عند وقوع حادث، و14% يوافقون غير ذلك، و10% محايداً.
- 62% يشعرون بالرضا العام عن طبيعة العمل وتصميمه الهندسي، و22% غير راضين عن التصميم الهندسي لطبيعة العمل، و16% محايداً.

- **فيما يختص بقيام مسئول السلامة بمهامه تجاه العاملين فكانت النتائج التالية**

- 70% يوافقون على أن يعمل مسئول السلامة بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي، و16% يوافقون على أن مسئول السلامة لا يعمل بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي و14% محايدًا.
- 50% يوافقون على أن مسؤولالسلامة يقوم بعقد ندوات ودورات تدريبية وورش عمل تتعلق بسلامة المنشأة والعاملين، و34% لا يوافقون على ذلك، و16% محايداً.
- 78% منهم يوافقون على أن مسؤول يساعد العاملين عند وقوع إصابة عمل وتوجيهم لنيل حقوقهم حسب النظم واللوائح، و12% لا يوافقون على ذلك، 10% محايداً.
- 76% يوافقون على التزام إدارة السلامة بالمصنع بتوفير كافة مستلزمات الوقاية الشخصية، و18% لايوافقون على ذلك، و6% محايداً.

- 58% يرون أنّ إدارة السلامة تلتزم بتفعيل كافة أنظمة السلامة بالمصنع، و22% لايرون ذلك و20% محايداً.
- 54% يشعرون بالرضا العام تجاه مسؤولي السلامة بالمصنع، و30% لايشعرون بالرضا تجاه مسؤولي السلامة بالمصنع، و16% محايداً.

- **فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي**

- 84% يقومون بالالتزام كاملاً بارتداء ملابس الوقاية الشخصية، و10% لايلتزمون بارتداء ملابس الوقاية الشخصية، و6% محايداً.
- 90% يلتزمون التزامًا تامًا باللوائح والقوانين التي تتعلق بسلامة العاملين بالمصنع، و10% محايداً.
- 82% يمكنهم أن يحسنوا التصرف عند وقوع حادث أثناء العمل، و2% لا يمكنهم أن يحسنوا التصرف عند وقوع حادث أثناء العمل، و16% محايداً.
- 82% منهم على دراية تامة بأنظمة السلامة داخل المصنع والورشة، و4% ليس على دراية تامة بتلك الأنظمة، و14% محايداً.

- **أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي**

- 82% تم إخضاعهم لفحصٍ طبي قبل مباشرة العمل، و4% لا تم إخضاعهم لفحص طبي قبل مباشرة العمل، و14% محايداً.
- 76% يتم إخضاعهم لفحصٍ طبي دوري، و16% لا يتم إخضاعهم لفحص طبي دوري، و8% محايداً.
- 42% يوافقون على تلقِّيهم لتدريبات في الإسعافات الأولية عند وقوع حادث أو إصابة عمل، و38% لم يوافقوا على ذلك، و20% محايداً.
- 70% يوافقون على أن مستوي التغذية كافية، و10% لا يوافقون على ذلك، و20% محايداً.
- 72% يوافقون على أنّ هنالك خدمات علاجية مستمرة، و12% لا يوافقون على ذلك، و16% محايداً.

المصنع(B)

وجدت الدراسة كثيرًا من المخاطر منها:

- السيور الناقلة.
- حركة الآليات.
- الكهرباء.
- الحريق.
- مرافق الإنتاج.

تقييم المخاطر للمصنع

الخطر	الاحتمالية	الشدة
السيور الناقلة	3	4
حركة الآليات	3	3
مخاطر الكهرباء	2	3
الحريق	4	5
مرافق الإنتاج	4	4

نتائج تحليل المصنع B

اتضح من خلال تحليل بيانات المصنع (B) من خلال الاستبانة المطروحة على عينة البحث ما يأتي:

فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ

- 40% يوافقون على أن مستوى الإضاءة مناسب جداً لطبيعة العمل، و30% منهم لا يوافقون عليها، و30% محايدين.
- 100% يوافقون على أن مستوى التهوية مناسب جداً لطبيعية العمل.
- 26% يوافقون على أن الضجيج لا يؤثر على سلامتهم، و36% أن الضجيج يؤثر على سلامتهم ، و38% محايداً.
- 76% يوافقون على أن مستوى الاهتزاز مناسب مع طبيعة العمل، و24% يوافقون على أن مستوي الاهتزاز غير مناسب مع طبيعة العمل، و12% منهم محايدًا.
- 76% يوافقون على أن درجة الحرارة مناسبة مع طبيعة العمل، و12% يوافقون على أن مستوى درجة الحرارة غير مناسبة مع طبيعة العمل، و12% محايدًا.
- 76% يوافقون على أن مكان العمل آمن من مخاطر الكهرباء، و24% محايدًا.
- 74% يوافقون على أن عند وقوع حادث يمكنهم التصرف معه، و26% محايداً.
- 76% يوافقون على أن عند وقوع حريق يمكنهم الوصول لأجهزة الإنذار والإطفاء في زمن قياسي، و24% محايداً.
- 86% يمكنهم التصرف والوصول السريع إلى مخارج الطوارئ عند وقوع حادث، و14% محايداً.
- 38% يشعرون بالرضا العام عن طبيعة العمل وتصميمه الهندسي، و12% غير راضين عن التصميم الهندسي لطبيعة العمل، و50% محايداً.

- **فيما يختص بقيام مسئول السلامة بمهامه تجاه العاملين فكانت النتائج التالية**

- 88% يوافقون على أن يعمل مسئول السلامة بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي، و12% محايدًا.
- 64% يوافقون على أن مسؤول السلامة يقوم بعقد ندوات ودورات تدريبية وورش عمل تتعلق بسلامة المنشأة والعاملين، و24% لا يوافقون على ذلك، و12% محايداً.
- 76% منهم يوافقون على أن مسؤول السلامة يساعد العاملين عند وقوع إصابة عمل وتوجيهم لنيل حقوقهم حسب النظم واللوائح، و12% لا يوافقون على ذلك، و12% محايداً.
- 76% يوافقون على التزام إدارة السلامة بالمصنع بتوفير كافة مستلزمات الوقاية الشخصية، و24% محايداً.
- 76% يرون أنّ إدارة السلامة تلتزم بتفعيل كافة أنظمة السلامة بالمصنع، و24% محايداً.
- 88% يشعرون بالرضا العام تجاه مسؤولي السلامة بالمصنع، و12% محايداً.

- **فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي**

- 100% يقومون بالالتزام كاملاً بارتداء ملابس الوقاية الشخصية.
- 88% يلتزمون التزاماً تامًا باللوائح والقوانين التي تتعلق بسلامة العاملين بالمصنع، و12% محايداً.
- 82% يمكنهم أن يحسنوا التصرف عند وقوع حادث أثناء العمل، و2% لا يمكنهم أن يحسنوا التصرف عند وقوع حادث أثناء العمل، و16% محايداً.

- 82% منهم على دراية تامة بأنظمة السلامة داخل المصنع والورشة، و4% ليسوا على دراية تامة بتلك الأنظمة، و14% محايداً.

- **أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي**

- 88% تم إخضاعهم لفحصٍ طبي قبل مباشرة العمل، و10% لم يتم إخضاعهم لذلك، و2% محايداً.
- 76% يتم إخضاعهم لفحصٍ طبي دوري، و16% لا يتم إخضاعهم لفحص طبي دوري، و8% محايداً.
- 42% يوافقون على تلقِّيهم لتدريبات في الإسعافات الأولية عند وقوع حادث أو إصابة عمل، و38% لم يوافقوا على ذلك، و20% محايداً.
- 28% يوافقون على أن مستوي التغذية كافية، و62% لا يوافقون على ذلك، و10% محايداً.
- 72% يوافقون على أنّ هنالك خدمات علاجية مستمرة، و12% لا يوافقون على ذلك، و16% محايداً.

المصنع C

نتائج تحليل المصنع(C)

اتضح من خلال تحليل بيانات المصنع (C) من خلال الاستبانة المطروحة على عينة البحث ما يأتي:

- **فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ**

- 90% يوافقون على أن مستوى الإضاءة مناسب جداً لطبيعة العمل، و10% محايدين.
- 100% يوافقون علي أن مستوى التهوية مناسب جداً لطبيعية العمل.
- 20% يوافقون على أن الضجيج لا يؤثر على سلامتهم 60% أن الضجيج يؤثر على سلامتهم و20% محايداً.
- 50% يوافقون على أن مستوى الاهتزاز مناسب مع طبيعة العمل، و10% يوافقون على أن مستوي الاهتزاز غير مناسب مع طبيعة العمل، و40% منهم محايداً.
- 70% يوافقون على أن درجة الحرارة مناسبة مع طبيعة العمل، و30% يوافقون على أن مستوى درجة الحرارة غير مناسبة مع طبيعة العمل.
- 70% يوافقون على أن مكان العمل آمن من مخاطر الكهرباء، و20% يوافقون على أن مكان العمل غير آمن من مخاطر الكهرباء، و10% محايداً.
- 70% يوافقون على أن عند وقوع حادث يمكنهم التصرف معه، و30% يوافقون على أنهم لايمكنهم التصرف عند وقوع حادث.
- 60% يوافقون على أن عند وقوع حريق يمكنهم الوصول لأجهزة الإنذار والإطفاء في زمن قياسي، و30% يوافقون على عدم وصولهم لأجهزة الإنذار والإطفاء في زمن قياسي، و10% محايدًا.
- 100% يوافقون على أن يمكنهم التصرف والوصول السريع إلى مخارج الطوارئ عند وقوع حادث.
- 50% يشعرون بالرضا العام عن طبيعة العمل وتصميمه الهندسي، و30% غير راضين عن التصميم الهندسي لطبيعة العمل، و20% محايدًا.

- **فيما يختص بقيام مسئول السلامة بمهامه تجاه العاملين فكانت النتائج التالية**

  - 50% يوافقون على أن يعمل مسئول السلامة بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي، و40% يوافقون على أن مسئول السلامة لا يعمل بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي و10% محايدًا.
  - 50% يوافقون على أن مسؤول السلامة يقوم بعقد ندوات ودورات تدريبية وورش عمل تتعلق بسلامة المنشأة والعاملين، و40% لا يوافقون على ذلك، و10%محايداً.
  - 60% منهم يوافقون على أن مسؤول يساعد العاملين عند وقوع إصابة عمل وتوجيهم لنيل حقوقهم حسب النظم واللوائح، و40% لا يوافقون على ذلك.
  - 80% يوافقون على التزام إدارة السلامة بالمصنع بتوفير كافة مستلزمات الوقاية الشخصية، و10% لا يوافقون على ذلك، و10%محايداً.
  - 50% يرون أنّ إدارة السلامة تلتزم بتفعيل كافة أنظمة السلامة بالمصنع، و40% لا يرون ذلك و10% محايداً.
  - 50% يشعرون بالرضا العام تجاه مسؤولي السلامة بالمصنع، و30% لا يشعرون بالرضا تجاه مسؤولي السلامة بالمصنع، و20%محايداً.

- **فيما يختص بوعي العاملين بأنظمة السلامةفقد تبين من النتائج ما يأتي**

  - 80% يقومون بالالتزام كاملاً بارتداء ملابس الوقاية الشخصية و20% لا يلتزمون بارتداء ملابس الوقاية الشخصية.
  - 80% يلتزمون التزاماً تامًا باللوائح والقوانين التي تتعلق بسلامة العاملين بالمصنع، و10% لايلتزمون التزاماً تامًا باللوائح والقوانين التي تتعلق بسلامة العاملين بالمصنع، و10% محايداً.
  - 60% يمكنهم أن يحسنوا التصرف عند وقوع حادث أثناء العمل، و20% لا يمكنهم أنيحسنوا .التصرف عند وقوع حادث أثناء العمل، و20% محايداً.
  - 80% منهم على دراية تامة بأنظمة السلامة داخل المصنع والورشة، و10% ليس على دراية تامة بتلك الأنظمة، و10% محايداً.

- **أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي**

  - 100% تم إخضاعهم لفحصٍ طبي قبل مباشرة العمل.
  - 80% يتم إخضاعهم لفحصٍ طبي دوري، و10% لا يتم إخضاعهم لفحص طبي دوري، و10% محايداً.
  - 40%ي وافقون على تلقّيهم لتدريبات في الإسعافات الأولية عند وقوع حادث أو إصابة عمل، و60% لم يوافقوا على ذلك.
  - 30% يوافقون على أن مستوي التغذية كافية، و70% لا يوافقون على ذلك.
  - 60% يوافقون على أنّ هنالك خدمات علاجية مستمرة، و30% لا يوافقون على ذلك، و10% محايداً.

المصنع D

المخاطر التي توجد في المصنع

- السيور الناقلة.
- حركة الآليات.
- الكهرباء.
- الحريق.
- مرافق الإنتاج

تقييم المخاطر للمصنع

الخطر	الاحتمالية	الشدة
السيور الناقلة	3	4
حركة الآليات	4	3
مخاطر الكهرباء	2	3
الحريق	1	5
مرافق الإنتاج	2	5

اتضح من خلال تحليل بيانات المصنع (D) من خلال الاستبانة المطروحة على عينة البحث ما يأتي:

- **فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ**

- 76% يوافقون على أن مستوى الإضاءة مناسب جداً لطبيعة العمل، و12% يوافقون علي أن مستوى الإضاءة غير مناسب لطبيعة العمل، و12% محايدين.
- 100% يوافقون على أن مستوى التهوية مناسب جداً لطبيعية العمل.
- 70% يوافقون على أن الضجيج لا يؤثر على سلامتهم، و30% محايداً.
- 88% يوافقون على أن مستوى الاهتزاز مناسب مع طبيعة العمل، و12% منهم محايداً.
- 88% يوافقون على أن درجة الحرارة مناسبة مع طبيعة العمل، و12% محايداً.
- 100% يوافقون على أن مكان العمل آمن من مخاطر الكهرباء.
- 100% يوافقون على أن عند وقوع حادث يمكنهم التصرف معه.
- 100% يوافقون على أن عند وقوع حريق يمكنهم الوصول لأجهزة الإنذار والإطفاء في زمن قياسي.
- 100% يوافقون على أن يمكنهم التصرف والوصول السريع إلى مخارج الطوارئ عند وقوع حادث.
- 82% يشعرون بالرضا العام عن طبيعة العمل وتصميمه الهندسي، و18% محايداً.

- **فيما يختص بقيام مسؤول السلامة بمهامه تجاه العاملين فكانت النتائج التالية**

- 82% يوافقون على أن يعمل مسئول السلامة بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي، و18%محايدًا.
- 100% يوافقون على أن مسؤول السلامة يقوم بعقد ندوات ودورات تدريبية وورش عمل تتعلق بسلامة المنشأة والعاملين.

- 100% منهم يوافقون على أن مسؤول يساعد العاملين عند وقوع إصابة عمل وتوجيهم لنيل حقوقهم حسب النظم واللوائح.
- 100% يوافقون على التزام إدارة السلامة بالمصنع بتوفير كافة مستلزمات الوقاية الشخصية.
- 100% يرون أنّ إدارة السلامة تلتزم بتفعيل كافة أنظمة السلامة بالمصنع
- 100% يشعرون بالرضا العام تجاه مسؤولي السلامة بالمصنع.

- **فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي**

- 100% يقومون بالالتزام كاملاً بارتداء ملابس الوقاية الشخصية.
- 82% يلتزمون التزاماً تامًا باللوائح والقوانين التي تتعلق بسلامة العاملين بالمصنع، و18% محايداً.
- 100% يمكنهم أن يحسنوا التصرف عند وقوع حادث أثناء العمل
- 76% منهم على دراية تامة بأنظمة السلامة داخل المصنع والورشة، و12% ليس على دراية تامة بتلك الأنظمة، و12% محايداً.

- **أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي**

- 100% تم إخضاعهم لفحصٍ طبي قبل مباشرة العمل.
- 70% يتم إخضاعهم لفحصٍ طبي دوري، و30%محايدًا.
- 88% يوافقون على تلقيهم لتدريبات في الإسعافات الأولية عند وقوع حادث أو إصابة عمل، و12%محايداً.
- 88% يوافقون على أن مستوي التغذية كافية، و12%محايداً.
- 100% يوافقون على أنّ هنالك خدمات علاجية مستمرة.

مناقشة النتائج

بالنسبة لأنظمة السلامة

بالنسبة للمصنع A

- فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ:

- 65% يوافقون على أن طبيعة المصنع موائمة لطبيعة العمل، و16% لا يوافقون على ذلك، و19% محايدًا، وهذا يعني أن طبيعة العمل بالمصنع جيدة الي حد ما وهذا يعني يحتاج المصنع إلى مرواح تهوية وعمل عوازل لمصادر الضجيج.

- فيما يختص بقيام مسئول السلامة بمهامه تجاه العاملين فكانت النتائج التالية:

- 64.3% يوافقون على أن مسؤول السلامة يقوم بمهامه تجاه العاملين، و22% يوافقون على أن مسؤول السلامة لا يقوم بمهامه تجاه العاملين، و13.6% محايداً، وهذا يعني أن مسؤول السلامة يقوم بمهامه إلى حد ما، وأن الدورات التدريبية غير كافية ويحتاج العاملين بالمنشأة إلى توعية بأنظمة السلامة.

- فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي:

- 84.5% يوافقون على وعي بأنظمة السلامة، و4% غير موافقين، و11.5% محايداً، وهذا يعني أنّ العاملين على وعي بأنظمة السلامة، مما يوضح أنّ أثر الثواب والعقاب كان إيجابياً في هذه الناحية، والذي يتمثل في الإنذارات والخصم والتحفيز.

- **أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي:**

- 68.4% يوافقون على مستوى الصحة المهنية في المصنع، و16% لايوافقون على مستوى الصحة المهنية في المصنع، و15.6% محايداً، وهذا يعني أنّ هذا المستوي يحتاج الي فحص دوري ورفع مستوى وعي العاملين بالإسعافات الأولية.

بالنسبة للمصنع B

- **فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ:**

- 66% يوافقون على أن طبيعة العمل موائمة لمعايير السلامة، و11% لايوافقون على ذلك، و23% محايدًا، مما يشير إلى أن طبيعة العمل موائمة لمعايير السلامة بصورة جيدة، وتحتاج إلى تحسين مستوى الإضاءة وعمل عوازل لمصادر الضجيج.

- **فيما يختص بقيام مسئول السلامة بمهامه تجاه العاملين فكانت النتائج التالية:**

- 78% يوافقون على أن مسؤول السلامة يقوم بمهامه تجاه العاملين، و16% لا يوافقون على أن مسؤول السلامة لا يقوم بمهامه تجاه العاملين، و6% محايدًا وهذا يعنى أن مسؤول السلامة يقوم بمهامه بصورة جيدة؛لأنه يقوم بنشر ثقافة الأمن والسلامة ورفع مستوى الوعي.

- **فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي:**

- 88% على وعي بأنظمة السلامة، و11.5% غير ذلك، و10.5% محايداً، وهذا يعني أن العاملين عل وعي بصورة جيدة جداً مما يدل على أن هناك تدريب للعاملين على أنظمة السلامة وقوانينها.

- **أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي:**

- 61.2% يوافقون على مستوى الصحة المهنية في المصنع، و27.6%غير موافقون على ذلك، و 11.2% محايد وهذا يعني أن مستوي التغذية سيئ جداً والتغذية في المصنع تكون في شكل بدلات وليس وجبات.

بالنسبة للمصنع C

- فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ:

- 68% يوافقون على أن طبيعة العمل موائم لمعاير السلامة، 21% لا يوافقون على ذلك، 11% محايداً، وهذا يعني أن طبيعة العمل في هذا المصنع جيدة ولكنها تحتاج إلى عزل مصدر الضجيج؛ لأن الضجيج في هذا المصنع عالٍ جداً، والاهتزاز يحتاج إلىحلٍ هندسي.

- فيما يختص بقيام مسؤول السلامة بمهامه تجاه العاملين فكانت النتائج التالية:

- 56.6% يوافقون على أنّ مسئول السلامة يقوم بمهامه، و25% لا يوافقون على ذلك، و10% محايداً.

- فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي:

- 75% على وعي بأنظمة السلامة، و15%غير ذلك، و10% محايداً وهذا يشير إلى أن مستوى وعي العاملين بأنظمة السلامة والتزامهم التزاماً تامًا باللوائح والقوانين التي تتعلق بسلامة العاملين بالمصنع بصورة جيدة،مما يدل على احتياج لمزيد من الدورات التدريبة.

- أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي:

- 62% يوافقون على مستوى الصحة المهنية، و20% لايوافقون على ذلك، و4% محايداً، وهذا يشير إلى تفاوتٍ بين متطلبات الصحة المهنية، فبينما مستوى التغذية ضعيف جدًا يحتاج إلى رفعٍ وبذل مزيد من الجهود لتحسينه، نجد أنّ مستوى الفحص الطبي الدوري ممتاز جداً.

- **بالنسبة للمصنع D**

- فيما يختص بطبيعة العمل ومواءمته لمعايير السلامة أظهرت النتائج أنّ:

- 90% يوافقون على طبيعة العمل بالمصنع، و10% محايداً، وهذا يعني أن طبيعة العمل ممتازة جدًا، ممّا يثبت أن المصنع مؤهل من هذه الناحية تأهيلاً ممتازًا للعمل.

- فيما يختص بقيام مسئول السلامة بمهامه تجاه العاملين فكانت النتائج التالية:

- 97% يوافقون على أن مسئول السلامة يقوم بمهامه، و3% محايدًا،وذلك هذا يؤكد على قيام مسؤولي السلامة بعمل دورات تدريبة للعاملين وتعريفهم بقوانين السلامة كاملة، وتقديم المساعدة لهم عند وقوع إصابة عمل وتوجيهم لنيل حقوقهم حسب النظم واللوائح، إضافة إلى عمق الصلات بينهم وقوتها.

- فيما يختص بوعي العاملين بأنظمة السلامة فقد تبين من النتائج ما يأتي:

- 89.5% على وعى بأنظمة السلامة، و3% غير ذلك، و7.5% ، وهذا يعني أن مستوى وعيهم بصورة ممتازة، لالتزامهم الكامل بارتداء ملابس الوقاية الشخصية وأيضاً، ولهم القدرة على التصرف عند وقوع حادث.

- أما بالنسبة للصحة المهنية وسلامة العاملين فقد أظهرت النتائج ما يأتي:

- 89.2% يوافقون على مستوى الصحة المهنية، و10.8% محايد وهذا يعنى أن مستوى الصحة المهنية جيد جداً، وذلك لوجود كوادر طبية (24) ساعة ووجود غرفة إنعاش ومركز صحي متكامل.

المصنع B

مناقشة نتائج تقييم المخاطر

بالنسبة للمصنع B

السيور الناقلة

- الاحتمالية

أُعْطِيَتْ السيور الناقلة الرقم (2) هذا يعني احتمالية وقوع الخطر ضعيف والسبب في ذلك وجود وقاية للسيور بمادة المطاط مما يشكل لها حماية كافية.

- الشدة

أُعْطِيَتْ السيور الناقلة الرقم (4) هذا يعني تأثير الخطر إذا وقع يكون كبيرًا مثل موت عامل / تدفق المادة المنقولة... الخ.

حركة الآليات

- الاحتمالية

أُعْطِيَتْ حركة الآليات الرقم (3) هذا يعني أن وقوع الخطر محتمل حدوثه بنسبة متوسطة والسبب في ذلك ان التحكم في اليات يتم التحكم فيه من قبل الإنسان الذي يحدث من قبله الخطأ احياناً.

- الشدة

أُعْطِيَتْ حركة الآليات الرقم (3) هذا يعني أنّ تأثير الخطر اذ وقع يكون أقصي شدة موت شخص واحد وأقل شدة عدم التأثر بالخطر.

مخاطر الكهرباء

- الاحتمالية

أُعْطِيَتْ مخاطر الكهرباء الرقم (2) وهذا يعني أنّ إحتمالية وقوع الخطر بصورة ضعيفة وذلك لوجود التوصيل الأرضي الذي يشكل حماية كافية.

- الشدة

أُعْطِيَتْ مخاطر الكهرباء الرقم (2) وهذا يعني أنّ التأثير يكون ضعيفًا عند وقوع الحدث؛ وذلك لنوع للتوصيلات الموصلة في المصنع.

الحريق:

- الاحتمالية

أُعْطِيَ الحريق الرقم (3) وهذا يشير إلى أنّ احتمالية وقوع الخطر بنسبة متوسطة؛ وذلك بطبيعة عمل المصنع والمواد التي توجد فيه.

- الشدة

أُعْطِيَ الحريق الرقم (5) وهذا يعني وقوع خسائر بشرية ومادية عند وقوع الحريق.

مرافق الإنتاج

- الاحتمالية

أُعْطِيَتْ مرافق الإنتاج الرقم (3) وهذا يعني أنّ احتمالية وقوع الحدث بصورة متوسطة وذلك لما موضوع من قوانين عمل للمصنع.

- الشدة

أُعْطِيَتْ مرافق الإنتاج الرقم (4) وهذا يعني تأثير الخطر يكون كبيرًا جدًا مثل تأثر الطاحونة التي تعمل على فقد وعي العامل إذا استمرّ العمل عليها لأكثر من نصف ساعة.

المصنع (D)

مناقشة نتائج تقييم المخاطر:

السيور الناقلة

- الاحتمالية:

أُعْطِيَتْ السيور الناقلة الرقم (2) وهذا يعني احتمالية وقوع الخطر؛ والسبب في ذلك لأنّ هنالك وجود صوت قبل تشغيل السير بخمسة دقائق، مما يشكِّل مصدر انتباه للعاملين بالمصنع إذا حدث أي عطل يوجد توقيف أوتوماتيك يضاف إلى ذلك تصريح العمل.

- الشدة

أُعْطِيَتْ السيور الناقلة الرقم (4) هذا يعني تأثير الخطر إذا وقع يكون كبيراً مثل موت عاملٍ أو تدفّق المادة التي تُحْمَل... الخ.

حركة الآليات

- الاحتمالية

أُعْطِيَتْ حركة الآليات الرقم (4) هذا يعني أنّ وقوع الخطر محتمل حدوثه بنسبة متوسطة والسبب في ذلك أنّ التحكم في الآليات يتم من قبل الإنسان الذي يحدث من قبله الخطأ أحيانا إضافة إلى ذلك أرضية المصنع غير مستوية مما يزيد نسبة الحوادث.

- الشدة

أُعْطِيَتْ حركة الآليات الرقم (3) هذا يعني أنّتأثير الخطر إذا وقع يكون أقصى شدةً موت شخصٍ واحدٍ وأقل شدةً عدم التأثر بالخطر.

مخاطر الكهرباء

- الاحتمالية

أُعْطِيَتْ مخاطر الكهرباء الرقم (2) هذا يعني أن احتمالية وقوع الخطر بصورة ضعيفة، وذلك لوجود التوصيل الأرضي الذي يشكل حماية كافية.

- الشدة

أُعْطِيَتْ مخاطر الكهرباء الرقم (2) وهذا يعني ان عند وقوع الحدث يكون التأثير ضعيفًا؛ وذلك لنوع التوصيلات الموصلة في المصنع.

الحريق

- الاحتمالية

أُعْطِي الحريق الرقم (1) هذا يعني أن احتمالية وقوع الخطر محتمل بنسبة ضعيفة جدًا؛ وذلك لوجود أجهزة إنذار مما يجعل إدراك الأمر سريعًا قبل وقوع الحريق، إضافة إلى تدريب 25% من العاملين على استخدام طفاية الحريق.

- الشدة

أُعْطِيَ الحريق الرقم (5) وهذا يعني حدوث خسائر بشرية ومادية عند وقوع الحريق.

مر افق الإنتاج

- الاحتمالية

أُعطيتْ مرافق الإنتاج الرقم (2) هذا يعني أن احتمالية وقوع الحادث بصورة ضعيفة؛ وذلك لقوانين العمل المصممة للمصنع، وتهيئة البيئة للعاملين في تلك المنطقة.

- الشدة

أُعْطِيتْ مرافق الإنتاج الرقم (4) هذا يعني تأثير الخطر يكون كبيرًا جدًا؛ مثل تأثّر الطاحونة التي تعمل على فقد وعي العامل إذا استمر العمل عليها أكثر من نصف ساعة.

الخاتمة

يخلص البحث إلى وجود اهتمام بأنظمة السلامة والصحة المهنية في المصانع التي أُجْرِيَتْ عليها الدراسة، وخير مثال: نظام السلامة في المصنع D فهو مؤهل جدًا ومن أفضل المصانع التي تهتم بالسلامة في الولاية. غير أنّ هذا الاهتمام لا يمثّل الصورة المطلوبة أو النموذجية، بل يحتاج إلى مزيد من الاهتمام، وبذل الجهود التطويرية في هذا المجال للارتقاء به، فقد كشفتْ الدراسة وجود تقصيرٍ في بعض نظم السلامة والصحة المهنية؛ مثل: التغذية في مصنع C،B. وكذلك طبيعة العمل غير مؤهلة في مصنع C.B.A. كذلك توصلت الدراسة الى وجود مخاطر منها: مخاطر السيور الناقلة التي قد تسبب في وفاة العاملين- مخاطر حركة الآليات- مخاطر الكهرباء التي تسبب في حريق، وفاة عاملين، تعطيل الاجهزة- مخاطر الحريق التي تسبب في وفاة العاملين ودمار المنشاة- مخاطر مرافق الإنتاج التي تسبب في الامراض المهنية.

التوصيات

توصي الدراسة بالآتي

- دراسة الاثر البيئي لتلك المصانع.
- دراسة الأمراض المهنية الناتجة من المصانع.
- تحسين مستوى الإضاءة في مصنع (B).
- عزل مواقع الضجيج بكل من مصنع D، C، B،A.
- رفع مستوى التغذية في مصنع (B).
- رفع مستوى التغذية في مصنع (C).

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Contribution of Exponential and Integral Functions in some Probability Distributions

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Abstract

This article reviews and interprets exponential and integral functions on representations of some probability distributions. The literature survey covers Poisson distribution, Normal Distribution, Exponential Distribution, Gamma Distribution, and Weibull Distribution. The article explained in detail the contribution of both functions on the chosen probability distributions. The most important recommendation for those who want to be good statisticians is that they must have a good background in mathematics.

Keywords: Exponential, Gamma, Beta, Contribution

مساهمة الدالة الأسية والدالة التكاملية في بعض التوزيعات الاحتمالية

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مُسْتَخْلَص

تناولت هذه الورقة دور الدالتين الأسية والتكاملية في التوزيعات الاحتمالية، غطي المسح النظري توزيع بواسون، التوزيع الطبيعي توزيع قاما وتوزيع ويبيل وتوضح بشيء من التفصيل مساهمة كل من الدالتين في انتاج تلك التوزيعات الاحتمالية المختارة، أهم التوصيات تحفيز الذين يرغبون في أن يكونوا إحصائيات بكفاءة عالية عليهم أن يحصلوا على خلفية مناسبة في الرياضيات.

كلمات مفتاحية: الأسية، قاما، بيتا، مساهمة

The overall Aim

The overall aim of this article is to display the contribution of exponential and integral functions in some probability distributions, and to stimulate those who want to be a good statistician i.e. they should be familiar with these functions as well as other functions.

Introduction

Statistics provides models that are needed to study situations involving uncertainties, in the same way as calculus provide models that are needed to describe, say Newton Laws of motions. The names which are connected most prominently with the growth of mathematical statistics are R.A. Fisher, J. Neyman, E.S. Peorson and A. Wald (John and Ronold, 1980, P:2). R.A. Fisher who got a degree of Doctor of Science at University of Evitcago in 1952 (George, 2014, P:1) , his name is connected with F distribution , George E.P.Box concluded in his article some questions about Fisher which might be asked :

- Was he an applied statistician?
- Was he a mathematical statistician?
- Was he data analyst?
- Was he a designer of investigation?

It is surely because he was all of these he was much more than the sum of the part . He provides an example we can seek to follow (George, 2014, P:2).Some of mathematical topics have useful background to statistics and probability distributions : They are Boolean algebra , calculus , functions , mathematical transforms , and matrices . (E.Frend & Waple, 1980, P:177) claimed that " The beginnings of the mathematics of statistics may be found in mid eighteenth – century studies in probability motivated by interest in games of chance. The theory thus developed for ″head or tails ″ or ″ red or black ″soon found applications in situations where the outcomes were ″boy or girl ″ or ″life or death″ or ″pass or fail″ and scholars began to apply probability theory to actuarial problems and some aspects of the social sciences and physics.

Literature of Some Probability Distributions

Exponential Functions

The origin of Exponential Function is the Binomial theorem is stated as:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Replacing x by $\frac{1}{n}$

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)\left(\frac{1}{n}\right)^2}{2!} + \frac{n(n-1)(n-2)\left(\frac{1}{n}\right)^3}{3!} + \cdots$$

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{\left(\frac{n}{n}\right)(1-\left(\frac{1}{n}\right))}{2!} + \frac{\left(\frac{n}{n}\right)(1-\left(\frac{1}{n}\right))(1-\left(\frac{2}{n}\right))}{3!} + \cdots$$

Taking the limit as $n \rightarrow \infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

= e (known as a magic number)

If we rase $\left(1+\frac{1}{n}\right)^n$ to x, then $\left(1+\frac{1}{n}\right)^{nx}$ and taking the limit as $n \rightarrow \infty$

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\left(\frac{1}{n}\right) + \frac{nx(nx-1)\left(\frac{1}{n}\right)^2}{2!} + \frac{nx(nx-1)(nx-2)\left(\frac{1}{n}\right)^3}{3!} + \cdots$$

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + \frac{\left(\frac{n}{n}\right)x}{1!} + \frac{\left(\frac{nx}{n}\right)(x-\left(\frac{1}{n}\right))}{2!} + \frac{\left(\frac{nx}{n}\right)(x-\left(\frac{1}{n}\right))(x-\left(\frac{2}{n}\right))}{3!} + \cdots$$

Taking the limit as $n \rightarrow \infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Which known as Exponential function (KA Stroud, 2004, P:4). The Exponential function plays important roles in many probability distributions such as: Poisson Distribution, Normal Distribution, Exponential Distribution and Weibull Distribution.

Poisson Distribution

Poisson distribution can be represented from the known Binomial distribution specifically, we shall investigate the limiting from the Binomial distribution when$n \rightarrow \infty$ and $p \rightarrow 0$, while np remains constant be λ , that is $np = \lambda$, and hence $p = \frac{\lambda}{n}$

That could be explained as follows:

$$b(x , n , p) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^{x}\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^{x}\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Then one of x factors should be divided in $\left(\frac{\lambda}{n}\right)^{x}$ into each factor of the product

$n(n-1)(n-2)\ldots\ldots(n-x+1)$ and write

$\left(1 - \frac{\lambda}{n}\right)^{n-x}$ as: $\left(\left(1 - \frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right)^{-\lambda}\left(1 - \frac{\lambda}{n}\right)^{-x}$

$$\frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{x-1}{n}\right)}{x!}(\lambda)^{x}\left(\left(1 - \frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right)^{-\lambda}\left(1 - \frac{\lambda}{n}\right)^{-x}$$

Finally, if we let $n \rightarrow \infty$ while $x$and λ remain fixed, we find that the limit becomes:

$$\frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{x-1}{n}\right)}{x!} \rightarrow 1$$

$$\left(1 - \frac{\lambda}{n}\right)^{-x} \rightarrow 1$$

$$\left(1 - \frac{\lambda}{n}\right)^{\frac{-n}{\lambda}} \rightarrow e$$

and hence, that the Limiting distribution becomes:

$$p(x , \lambda) = \frac{\lambda^{x}e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, \ldots.$$

which is a Poisson distribution this distribution, had been named after the French mathematician Simeon Poisson (1781 – 1840) (Susan and Jesse, 2003, P:9)

The Integral Functions

Some function are most conveniently defined in the form of integrals such as Gamma function and Beta function (Stroud, 2004, P:125)

The Gamma Function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \text{ and is convergent for } x > 0$$

If we replace $x \; by \; x + 1$

$$\text{Then } \Gamma(x+1) = \int_0^\infty t^{x} e^{-t} \, dt$$

Integrating by parts:

$$\Gamma(x+1) = \left[t^x \left(\frac{e^{-t}}{-1} \right) \right]_0^\infty + x \int_0^\infty e^{-t} \; t^{x-1} \, dt$$

$$= [0 - 0] + x\,\Gamma(x)$$

$$\therefore \; \Gamma(x+1) = x\,\Gamma(x)$$

Put $n = x$

$$\Gamma(n+1) = n\,\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$= (n-1)(n-2)\,\Gamma(n-2)$$

$$= (n-1)(n-2) \; \dots \; 1\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty t^0 e^{-t} \, dt = -\left[e^{-t} \; \right]_0^\infty = 0 + 1 = 1$$

$$\Gamma(n+1) = n(n-1)(n-2) \times 1 = n!$$

$$\therefore \; \Gamma(n+1) = n!$$

For Example: $\Gamma(7) \;=\; 6! \;=\; 720$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

The Beta Function

The Beta function $B(m\,,n\,)$ is defined by :

$$B(m\,,n\,) = \int_0^1 x^{m-1}(1\text{-}x\,)^{n-1}\;dx$$

which converges for $m > 0$ and $n > 0$

putting $(1 - x) = \mu \quad \therefore x = 1 - \mu$

$$dx \;=\; -\;d\mu$$

Limits when $x = 0\,, \mu = 1\,, when\; x = 1\,, \mu = 0$

$$B(m\,,n\,) = -\int_1^0 (1\text{-}\mu)^{m-1}\mu^{n-1}\;d\mu \;=\; \int_0^1 (1\text{-}\mu)^{m-1}\mu^{n-1}\;d\mu$$

$$= B(n\,,m\,)$$

$$\therefore B(m\,,n\,) = \;B(n\,,m\,)$$

The Relation between gamma and beta functions:

$$B(m\,,n\,) = \frac{\Gamma(m)\;\Gamma(n)}{\Gamma(m+n)}$$

The Normal distribution

A random variable has a normal distribution and it is referred to as a normal random variable if and only if its probability density is given by: (John and Ronold, 1980, P:5)

$$N(x\,,\mu\,,\sigma\,) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x\text{-}\mu}{\sigma}\right)^2} \text{for } -\infty < x < \infty \;\text{ where } \sigma > 0$$

Where σ is the standard deviation and μ is the mean of the distribution, we need to show that the total area from $-\infty\; to\; \infty\; is\; 1$, making the substitution $z = \frac{x-\,\mu}{\sigma}$ we get:

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{1}{2}z^2}dz$$

$$\text{Since} \int_0^{\infty} e^{-\frac{1}{2}z^2}\,dz = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\text{Then} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{\pi}}{\sqrt{2}} = 1$$

Moment Generating Function

The exponential function plays an impartment role is evaluating the moment generating function for the probability distribution, the two expectations$E[x] and\ E[x^2]$ are very useful in determine the mean and the variance of the distributions.

Definition

Let x be a random variable with Discrete probability distribution the moment generating function for x is denoted by: $m_x(t) = E[e^{tx}]$

Gamma Distribution

The theoretical basis for gamma distribution is the gamma function (KA Stroud,2004,P:136)

Definition

A random variable x with density:

$$f(x) = \frac{1}{\Gamma(\propto)\beta^n} x^{\propto-1} e^{-x/\beta}$$

$$x > 0 , \propto > 0 , \beta > 0$$

is said to have a gamma distribution with parameter $\propto$ and β with moment generating as :

$$m_x(t) = (1\text{-}\beta t)^{-\propto} t > \frac{1}{\beta}$$

and mean = $\propto \beta$, variance = $\propto \beta^2$

proof:

by definition:

$$m_x(t) = E[e^{tx}]$$

$$= \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\propto)\beta^{\propto}} x^{\propto-1} e^{-x/\beta}\, dx \qquad (1)$$

$$= \frac{1}{\Gamma(\propto)\beta^{\propto}} \int_{0}^{\infty} x^{\propto -1} e^{-(\frac{1}{\beta} - t)x} \quad dx \qquad (2)$$

$$Let\ z = (1 - \beta t)\frac{x}{\beta}$$

$$x = \beta z(1 - \beta t)$$

$$dx = \beta dz(1 - \beta t) \qquad (3)$$

Substituting in (2)

$$\mathrm{m_x(t)} = \frac{1}{\Gamma(\propto)\beta^{\propto}} \int_{0}^{\infty} (\frac{\beta z}{1 - \beta t})^{\propto -1} \frac{e^{-z}\beta dz}{(1 - \beta t)}$$

$$= \frac{1}{\Gamma(\propto)\beta^{\propto}} \frac{\beta^{\propto}}{(1 - \beta t)^{\propto}} \int_{0}^{\infty} z^{\propto -1} e^{-z} dz \qquad (4)$$

$$Sice: \Gamma(\propto) = \int_{0}^{\infty} z^{\propto -1} e^{-z} dz$$

$$\text{Then: } \mathrm{m_x(t)} = \frac{1}{\Gamma(\propto)(1 - \beta t)^{\propto}} \times \Gamma(\propto)$$

$$= (\frac{1}{1 - \beta t})^{\propto} = (1 - \beta t)^{-\propto}$$

$t < \frac{1}{\beta}$ to avoid division by zero

To find then mean:

$$u = \mathrm{E}[x] = \frac{d}{dt}(\mathrm{m_x(t)})$$

$$= \frac{d}{dt}(1 - \beta t)^{-\propto}{}_{\mathrm{t=0}}$$

$$= -\propto (1 - \beta t)^{-(\propto +1)} \times -\beta$$

$$= \propto \beta$$

$$var(x) = \mathrm{E}[x^2] - E[x]^2$$

$$E[x^2] = \frac{d^2}{dt^2}\left(m_x(t)\right) = \frac{d^2}{dt^2}\ (1 - \beta t)^{-\propto}{}_{t=0}$$

$$= \propto (\propto +1)\ (1 - \beta t)^{-\propto -2} \times -\beta_{t=0}$$

$$= \propto (\propto +1)\ \beta^2$$

$$var(x) = \propto (\propto +1)\ \beta^2 - (\propto \beta)^2$$

$$= \propto^2 \beta^2 + \propto \beta^2 - \propto^2 \beta^2 = \propto \beta^2$$

Hence the mean and variance of gamma distribution as:

$$\mu = \propto \beta \ and \ var(x) = \propto \beta^2$$

The Beta Distribution

Definition

A random variable X has a beta distribution and it is referred to as beta random variable if and only if:

$$f(x) = \begin{cases} \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} x^{\propto -1}(1-x)^{\beta -1} & \text{for} \quad 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

We need to show that:

$$\int_0^1 \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} x^{\propto -1}(1-x)^{\beta -1} \ dx = 1$$

That is:

$$\int_0^1 x^{\propto -1}(1-x)^{\beta -1} \ dx = \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\propto +\beta)}$$

$$= \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \int_0^1 x^{\propto -1}(1-x)^{\beta -1} \ dx$$

$$= \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \times \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\propto +\beta)} = 1$$

The mean and the variance of beta distribution:

$$the\ mean: \ \mu = \frac{\alpha}{\propto + \beta}$$

$$\sigma^2 = \frac{\alpha\beta}{(\propto + \beta)^2(\propto + \beta + 1)}$$

By definition: The mean is

$$\mu = \mathrm{E}[x]$$

$$= \frac{\Gamma(\propto + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 x^{\propto - 1}(1-x)^{\beta - 1}\ \mathrm{dx}$$

$$= \frac{\Gamma(\propto + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 x^{\propto - 1}(1-x)^{\beta - 1}\ \mathrm{dx}$$

Since

$$\int_0^1 x^{\propto - 1}(1-x)^{\beta - 1}\ \mathrm{dx} = B(\propto, \beta)$$

and

$$B(\propto + 1, \beta) = \frac{\Gamma(\alpha + 1)\,\Gamma(\beta)}{\Gamma(\propto + \beta + 1)}$$

$$= \frac{\alpha\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\propto + \beta + 1)}$$

$$\mu = \frac{\Gamma(\propto + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 x^{\propto - 1}(1-x)^{\beta - 1}\ \mathrm{dx} = \frac{\Gamma(\alpha + 1)\,\Gamma(\beta)}{\Gamma(\propto + \beta + 1)} \times \frac{\Gamma(\propto + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

$$\therefore\ \mu = \frac{\Gamma(\propto + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \times \frac{\alpha\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\propto + \beta)(\propto + \beta)} = \frac{\alpha}{\propto + \beta}$$

$$\sigma^2 = \mathrm{E}[x^2] - \mathrm{E}[x]^2$$

$$\mathrm{E}[x^2] = \int_0^1 \frac{\Gamma(\propto + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\propto + 1}(1-x)^{\beta - 1}\ \mathrm{dx}$$

$$Since: \int_0^1 x^{\propto + 1}(1-x)^{\beta - 1}\ \mathrm{dx} = B(\propto + 2, \beta)$$

$$and\ B(\propto +2\ ,\beta\ ) = \frac{\Gamma(\alpha+2)\ \Gamma(\beta)}{\Gamma(\propto +2+\beta)}$$

$$= \frac{(\alpha+1)\ \Gamma(\alpha+1)\ \Gamma(\beta)}{(\propto +\beta+1)\ \Gamma(\propto +\beta+1)}$$

$$= \frac{\alpha(\alpha+1)\ \Gamma(\alpha)\ \Gamma(\beta)}{(\propto +\beta+1)(\propto +\beta)\ \Gamma(\propto +\beta)}$$

$$\mathrm{E}[x^2] = \frac{\Gamma(\propto +\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \times \frac{\alpha(\alpha+1)\ \Gamma(\alpha)\ \Gamma(\beta)}{(\propto +\beta+1)(\propto +\beta)\ \Gamma(\propto +\beta)} = \frac{\alpha(\alpha+1)}{(\propto +\beta+1)(\propto +\beta)}$$

$$\sigma^2 = E[x^2] - E[x]^2 = \frac{\alpha(\alpha+1)}{(\propto +\beta+1)(\propto +\beta)} - \frac{\propto^2}{(\propto +\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\propto +\beta) - \propto^2 (\propto +\beta+1)}{(\propto +\beta+1)(\propto +\beta)^2}$$

$$= \frac{\propto^3 + \propto^2 \beta + \propto^2 + \propto \beta - \propto^3 - \propto^2 \beta - \propto^2}{(\propto +\beta+1)(\propto +\beta)^2}$$

$$\therefore \sigma^2 = \frac{\alpha\beta}{(\propto +\beta)^2(\propto +\beta+1\ )}$$

The mean and variance of beta distribution are:

$$\mu = \frac{\alpha}{\propto +\beta}\ \ and\ \sigma^2 = \frac{\alpha\beta}{(\propto +\beta)^2(\propto +\beta+1\ )}$$

The Weibull Distribution

Definition

A random variable x is said to have a Weibull distribution with parameter $\alpha\ and\ \beta$ if its density is given by: (Stroud, 2004, P:129)

$$f(x) = \propto \beta\ x^{\beta-1} e^{-\propto x^{\beta}}\ , x > 0\ , \propto > 0\ , \qquad \beta > 0$$

The mean of distribution is:

$$\mu = \alpha^{-\frac{1}{\beta}}\ \Gamma(1+\frac{1}{\beta})$$

Proof :

$$\mu = \mathrm{E}[x]$$

$$= \int_0^{\infty} x\ \propto \beta\ x^{\beta-1} e^{-\propto x^{\beta}}\ \mathrm{dx}$$

$$= \int_{0}^{\infty} \propto \beta \; x^{\beta} e^{-\propto x^{\beta}} \; \text{dx}$$

$$Let: \quad z = \propto x^{\beta}$$

$$x = (\frac{z}{\alpha})^{\frac{1}{\beta}}$$

$$dx = \frac{1}{\beta}(\frac{z}{\alpha})^{\frac{1}{\beta}-1} dz$$

$$Then: \quad \text{E}[x] = \int_{0}^{\infty} \propto \beta \; (\frac{z}{\alpha}) \; e^{-z}(\frac{1}{\alpha\beta})(\frac{z}{\alpha})^{\frac{1}{\beta}-1} \; \text{dz}$$

$$= \int_{0}^{\infty} (\frac{z}{\alpha})^{\frac{1}{\beta}} e^{-z} \; \text{dz}$$

$$= \alpha^{-\frac{1}{\beta}} \int_{0}^{\infty} z^{\frac{1}{\beta}} e^{-z} \; \text{dz}$$

$$Since: \int_{0}^{\infty} z^{\frac{1}{\beta}} e^{-z} \; \text{dz} \quad = \quad \Gamma(1 + \frac{1}{\beta})$$

$$\therefore \; \text{E}[x] = \mu = \alpha^{-\frac{1}{\beta}} \, \Gamma(1 + \frac{1}{\beta})$$

$$\sigma^2 = \text{E}[x^2] - \text{E}[x]^2$$

$$\text{E}[x^2] = \int_{0}^{\infty} x^{2} \propto \beta \; x^{\beta-1} e^{-\propto x^{\beta}} \; \text{dx}$$

$$= \int_{0}^{\infty} \propto \beta \; x^{\beta+1} e^{-\propto x^{\beta}} \; \text{dx}$$

$$Let: \; z = \propto x^{\beta}$$

$$x = (\frac{z}{\alpha})^{\frac{1}{\beta}}$$

$$dx = (\frac{1}{\alpha\beta})(\frac{z}{\alpha})^{\frac{1}{\beta}-1} dz$$

$$E[x^2] = \int_0^{\infty} \propto \beta \, (\frac{z}{\alpha})^{1+\frac{1}{\beta}} \frac{e^{-z}}{\alpha\beta} (\frac{z}{\alpha})^{\frac{1}{\beta}-1} \quad dz$$

$$= \int_0^{\infty} (\frac{z}{\alpha})^{\frac{2}{\beta}} e^{-z} \; dz$$

$$= (\frac{1}{\alpha})^{\frac{2}{\beta}} \int_0^{\infty} z^{\frac{2}{\beta}} e^{-z} \; dz$$

$$= \alpha^{-\frac{2}{\beta}} \Gamma\left(\frac{2}{\beta}+1\right)$$

$$\sigma^2 = E[x^2] - [x]^2$$

$$= \alpha^{-\frac{2}{\beta}} \Gamma(\frac{2}{\beta}+1) - \left(\alpha^{-\frac{1}{\beta}}(\Gamma 1 + \frac{1}{\beta})\right)^2$$

$$\therefore \sigma^2 = (\frac{1}{\alpha})^{\frac{2}{\beta}} \Gamma(\frac{2}{\beta}+1) - (\frac{1}{\alpha})^{\frac{2}{\beta}} \left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2$$

Discussion

The origin of Exponential function, Gamma and beta functions has been introduced. Grasping those functions helps to understand the behaviour of the probability distributions. Looking at Poisson distributions it could be seen that it is impossible to determine the probability function without the exponential function. The Poisson distribution function is written as: (Sanders and Smidt, 2000, pp 177-178)

$$p\,(x\,,\lambda\,) = \frac{\lambda^x e^{-\lambda}}{x!} \; for \; x = 0\,,1\,,2\,,3, \ldots$$

Some authors tended to evaluate $e^{-\lambda}$ to make life easy: They constructed table for e^{x} and e^{-x} one of the mis Frenund in his book (John and Ronold, 1980, P:16)

The gamma function reduces a lot of work specially in Integrating some difficult integral of the from:

$$\int_{0}^{\infty} t^{n} e^{-t} \ \mathrm{dt}$$

and also, the evaluation of $\Gamma(\frac{1}{2})$ which has been use in proving the function of standard normal function which severs as a probability density function.

The Beta function involved in the Beta distribution function,helps to find the mean and the variance of the distribution.

Conclusion

In this article, the exponential and integral functions have been displayed.

The roles in some probability distributions have been explained. Those who intend to be good statisticians are recommended to have good background in mathematics.

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Applications of Orlicz-Lorentz Spaces in Gateaux Differentiability

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Abstract

This study aimed to study Gateaux differentiability of the functional $\varphi_{\omega,\emptyset}(f) = \int_0^\infty \emptyset(f^*)\omega$ and of the Luxemburg norm, it followed the descriptive method and the study found that we can obtain the one-sided Gateaux derivatives in both cases by characterizing those points where the Gateaux derivative of the norm exists, we obtain a characterization of best $\varphi_{\omega,\emptyset}$-approximants from convex closed subsets, there a relation between best $\varphi_{\omega,\emptyset}$ -approximants and best approximants from a convex set.

Keywords: Gateaux Derivatives, Orlicz-Lorentz space, Best approximants

تطبيقات فضاءات اورليش-لورينتز في تفاضل مشتقة جيوتكس

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مُسْتَخْلَص

هدفت هذه الدراسة إلى دراسة مشتقة جيتوكس للدالى $\boldsymbol{\varphi}_{\omega,\emptyset}(\boldsymbol{f}) = \int_0 \emptyset(\boldsymbol{f}^*)$ ولنظيم لكسمبورج و اتبعت الدراسة المنهج الوصفي وتوصلت إلى إن بالإمكان الحصول على مشتقة جيتوكس ذات الجانب الواحد في كل الأحوال بتوصيف هذه النقاط التي تمثل نظيم خروج، وتم الحصول على توصيف للتقريب الأفضل $\boldsymbol{\varphi}_{\omega,\emptyset}$ من مجموعات جزئية محدبة مغلقة، هناك علاقة بين التقريب الأفضل ل $\boldsymbol{\varphi}_{\omega,\emptyset}$ و التقريب الافضل من مجموعة محدبة.

كلمات مفتاحية: مشتقة جيتوكس، فضاء أورليش- لورنتز، أفضل تقريب

Introduction

Let M_0 be the class of all real extended μ-mmeasurable functions on$[\,0,\alpha), 0 < \alpha \leq \infty$, where μ is the Lebesgue measure. As usual, for $f \in M_0$we denote its distribution function by $\mu_{f(\Omega)} = \mu(\{0 \leq x < \alpha: |f(x)| > \Omega\})(\Omega \geq 0)$, and its decreasing rearrangement by

$f^*(1+2\epsilon) = \inf\{\Omega: \mu_f(\lambda) \leq 1+2\epsilon\}\left(\epsilon \geq \frac{-1}{2}\right)$.

If two functions f and g have the same distribution functions we say they are equimeasurable and we denote it by $f \sim g$.

For other properties of μ_f and f^*, (see Bennet and Sharpley 1988, pp.36-42).

Let $\emptyset: R_+ \to R_+$be differentiable,convex,$\emptyset(0) = 0, \emptyset(1+2\epsilon) > 0$ for $\epsilon > \frac{-1}{2}$ and let

$\omega : (0,\alpha) \to (0,\infty)$ be aweight function, non-increasing and locally integrable.

If $\alpha = \infty$,we assume$\lim_{1+2\epsilon\to\infty} \omega(1+2\epsilon) = 0$ and $\int_0^\infty w(1+2\epsilon)d\mu(1+2\epsilon) = \infty$.

For $f \in M_0$let$\varphi_{\omega,\emptyset}(f) = \int_0^\alpha \emptyset\big(f^*(1+2\epsilon)\big)\omega(1+2\epsilon)\ d\mu(1+2\epsilon)$

In (Hudzik, Kamińska and Mastylo 2002 and Kamińska 1990 –Kamińska 1991), several authors studied geometric properties of the regular Orlicz-Lorentz spaces,

$\{f \in M_0: \varphi_{\omega,\emptyset}(\lambda f) < \infty \text{ for some} \Omega > 0\}$. The main ojective of this section is to study differentiability properties in the following subspace

$$\Lambda_{\omega,\emptyset} := \{f \in M_0: \varphi_{\omega,\emptyset}(\Omega f) < \infty \text{ for all} \Omega > 0\},$$

which appears to be convenient for our purpose. Under the norm given by

$$\|f\|_{\omega,\emptyset} = \inf\left\{\epsilon > 0: \varphi_{\omega,\emptyset}\left(\frac{f}{\epsilon}\right) \leq 1\right\},$$

$\Lambda_{\omega,\emptyset}$ is a Banach space (Kamińska 1990). It is clear that if ω is constant, $\Lambda_{\omega,\emptyset}$ becomes a subspace of finite elements $L_\emptyset^\infty$of the Orlicz space $L_\emptyset$ (see Rao and Ren 1991). On the other hand, setting $\emptyset(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}, 0 \leq \epsilon < \infty$, we obtain the Lorentz space $L_{(\omega,1+\epsilon)}$and

$\varphi_{\omega,\emptyset}(f) = \|f\|_{(\omega,1+\epsilon)}^{1+\epsilon}$. These weighted Lorentz spaces as a generalization of Lorentz space have been studied in (Halperin 1953).

If $\omega(1+2\epsilon) = \left(\frac{1+\epsilon}{1-\epsilon}\right)(1+2\epsilon)^{\left(\frac{1-\epsilon}{1+\epsilon}\right)-1}, 0 \leq -\epsilon \leq \epsilon < \infty$,

a good reference for a description of these spaces in $L(1+\epsilon, 1-\epsilon)$spaces in (Hunt 1966).A function $\sigma: [0,\alpha) \to [0,\alpha)$is called a measure preserving transformation (m.p.t) if for each μ-

measurable set $I \subset [0,\alpha), \sigma^{-1}(I)$ is μ-measurable and $\left(\sigma^{-1}(I)\right) = \mu(I)$. It is very important to emphasize that any m.p.t induces equi-measurability, that is, if $g \in M_0$ then $|g| \circ \sigma$ is a μ-measurable function on $[0,\alpha)$ and $|g| \circ \sigma \sim |g|$. For $g \in M_0$, we denote $\text{supp}(g) \coloneqq \{0 \leq x < \alpha : g(x) \neq 0\}$. In view of the assumptions on the weight ω, if $f \in \Lambda_{\omega,\emptyset}$, then $\lim_{1+2\epsilon\to\infty} f^*(1+2\epsilon) = 0$. In consequence, by Ryff's Theorem (Bennet and Sharpley, 1988) there is an m.p.t $\sigma: \text{supp}(f) \to \text{supp}(f^*)$ such that

$$|f| = f^* \circ \sigma \quad \mu - \text{a.e. on } \text{supp}(f) (1.1)$$

Moreover, if $\sigma: \text{supp}(f) \to \text{supp}(f^*)$ is any m.p.t fulfilling (1.1), then

$$\varphi_{\omega,\emptyset}(f) = \int_{\text{supp}(f)} \omega(\sigma) \emptyset(|f|)\, d\mu.$$

In fact, since $\emptyset(f^*)w \sim \emptyset(|f|)w(\sigma)$, so their integrals are equal (see Bennet and Sharpley 1988). Let $T: \Lambda_{\omega,\emptyset} \to R$ be a functional. For $f, h \in \Lambda_{\omega,\emptyset}$, we will use in this work the one-sided Gateaux derivatives $\gamma_T^+(f,h) = \lim_{1-2\epsilon\to 0^+} \frac{T(f+h(1-2\epsilon))-T(f)}{1-2\epsilon}$ and

$\gamma_T^-(f,h) = \lim_{1-2\epsilon\to 0^-} \frac{T(f+h(1-2\epsilon))-T(f)}{1-2\epsilon}$. (Carothers *et al.*, 1993) showed that if $\alpha = \infty, \emptyset(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}$, $0 < \epsilon < \infty$, and ω is astrictly decreasing function, then

$\gamma^+_{\varphi_{\omega,\emptyset}}(f,h) = (1+\epsilon)\int_0^\infty \omega\left(\tau_{f,h}\right)|f|^\epsilon(1-2\epsilon)g(f)h\, d\,\mu$, where $\left(\tau_{f,h}\right)$ is defined by

$$\begin{aligned}\left(\tau_{f,h}\right)(x) = \mu_f(|f(x)|) \\ &+ \mu\left(\left\{y : |f(y)| = |f(x)| \text{and } h(y)(1-2\epsilon)g\left(f(y)\right) > h(x)(1-2\epsilon)g\left(f(x)\right)\right\}\right) \\ &+ \mu\left(\left\{y : |f(y)| = |f(x)|, h(y)(1-2\epsilon)g\left(f(y)\right) = h(x)(1-2\epsilon)g\left(f(x)\right) \text{and } y \right.\right. \\ &\left.\left. \leq x\right\}\right). \quad (1.2)\end{aligned}$$

It is known that $\tau_{f,h}$ is an m.p.t and $|f| = f^* \circ \tau_{f,h}$ μ-a.e. on $\text{supp}(f)$ (see Ryff 1970).

In one sided Gateaux derivatives in $\Lambda_{\omega,\emptyset}$, we generalize this result. Usinga technique similar to that in (Levis and Cuenya, 2004, Theorem 2.6), we compute the one-sided Gateaux derivative of the modular for $0 < \alpha \leq \infty, \omega$ a non-increasing function, and $\emptyset$, a convex function. Here, we need to work with a suitable m.p.t. Also, we obtain the one-sided Gateaux derivative for the norm $\|\cdot\|_{\omega,\emptyset}$, called also the Luxemburg norm.

We say that $f \in \Lambda_{\omega,\emptyset}$ is a smooth point for T if there exists the Gateaux derivative of the functional T in f, i.e if $\gamma_T^+(f,h) = \gamma_T^-(f,h)$ for all $h \in \Lambda_{\omega,\emptyset}$ and we denote it by $\gamma_T(f,h)$. The set of smooth points for the functional $\varphi_{\omega,\emptyset}$ was investigated in (Levis and Cuenya, 2004).

Let $K \subset \Lambda_{\omega,\emptyset}$and $f \in \Lambda_{\omega,\emptyset}$be given, and considerthe problem of finding $h^{\bullet} \in K$ such that

$$T(f-h^{\bullet}) = \inf_{h \in K} T\,(f-h) =: E_T\,(f,K). \qquad (1.3)$$

Denote by $P_T(f,K)$the set of all$h^{\bullet} \in K$ fulfilling (1.3). Each element of$P_T(f,K)$ will be called the best T-approximant of f from K. If $T$is the Luxemburg norm, we only say the best approximant from K. Let $\alpha = 1$, let $\emptyset(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}$with $0 \le \epsilon < \infty$, let f be a simple function in $\Lambda_{\omega,\emptyset}$, and let $K \coloneqq \{g \in \Lambda_{\omega,\emptyset} : g \text{ is constant}\}$.

In (Levis and Cuenya, 2004), we give a characterization of the best,$\varphi_{\omega,\emptyset}$- approximants of f from K and we show the way to obtain the best $\varphi_{\omega,\emptyset}$-approximants maximum and minimum, which will be denoted by$\overline{f}$ and f respectively.We give a characterization of the best $\varphi_{\omega,\emptyset}$-approximants of$f \in \Lambda_{\omega,\emptyset}$ from a convex closed set, K, and we establish a relation between the best$\varphi_{\omega,\emptyset}$-approximants and the best approximants from K.Finally, we give a characterization of the best constant $\varphi_{\omega,\emptyset}$-approximants and we calculate the best constant$\varphi_{\omega,\emptyset}$-approximants maximum and minimum (Levis and Cuenya, 2007).

2.0 One sided Gateaux derivatives in $\Lambda_{\omega,\emptyset}$

We let $f, h \in \Lambda_{\omega,\emptyset}$for each $\epsilon \ge \frac{1}{2}$, we consider any m.p.t.

$\sigma_{f+h(1-2\epsilon)}$: $\operatorname{supp}\big(f+h(1-2\epsilon)\big) \to \operatorname{supp}\big(\big(f+h(1-2\epsilon)\big)^*\big)$ such that

$|f+h(1-2\epsilon)| = \big(f+h(1-2\epsilon)\big)^* \circ \sigma_{f+h(1-2\epsilon)}$-a.e.on $\operatorname{supp}\big(f+h(1-2\epsilon)\big)$.

In (Levis and Cuenya 2004), we showed that $\lim_{1-2\epsilon \to 0} \sigma_f\,(x) = \sigma_{f+h(1-2\epsilon)}\mu$-a.e on $E(f) \cap \operatorname{supp}(f)$ where $E(f) \coloneqq \{0 \le x < \alpha : \mu\{|f| = |f(x)|\} = 0\}$.

However, we give an example which shows that this result does not hold on the whole $\operatorname{supp}(f)$ (Levis and Cuenya 2007).

Example 2.1 For $\alpha = 1$, let $f = 2_{\chi_{[0,\frac{1}{2})}} + \chi_{[\frac{1}{2},1)}$and $h = \chi_{[\frac{1}{4},\frac{3}{4})}$.For $\frac{1}{2} < \epsilon < 0$,

we consider the m.p.t. defined by

$$\sigma_{(f+h(1-2\epsilon))}(x) = \left(x+\frac{1}{4}\right)\chi_{[0,\frac{1}{4})} + \left(x-\frac{1}{4}\right)\chi_{[\frac{1}{4},\frac{1}{2})} + x\chi_{[\frac{1}{2},1)},$$

and for$\frac{1}{2} < \epsilon < 0$ the m.p.t. defined by

$$\sigma_{(f+h(1-2\epsilon))}(x) = x\chi_{[0,\frac{1}{2})} + \left(x+\frac{1}{4}\right)\chi_{[\frac{1}{2},\frac{3}{4})} + \left(x-\frac{1}{4}\right)\chi_{[\frac{3}{4},1)}.$$

We observe that for all $0 \le x < 1, \lim_{1-2\epsilon \to 0} \sigma_{(f+h(1-2\epsilon))}(x)$does not exist.

Now our purpose is to define a sequence of m.p.t.$\sigma_{f+(1-2\epsilon)_n h}$

such that $\lim_{(1-2\epsilon)_n \to 0} \sigma_{f+(1-2\epsilon)_n h}\,(x)$exist for $x \in \text{supp}(f) \cup \text{supp}(h)$.To prove it we need some auxiliary lemmas.

Lemma 2.2 Let $R \subset [0,\alpha)$be a μ-measurable set with $\mu(R) = b > 0$.

Then $\sigma\colon [\,0,\alpha) \to [0,b)$defined by $\sigma(x) = \mu(R \cap [0,x])$is a non-decreasing continuous function with $\sigma(0) = 0$ and $\lim_{x\to\alpha^-} \sigma(x) = b$.

Lemma 2.3 Let σ be the function given in Lemma 2.2. Then$\sigma\colon R \to [0,b)$ is an m.p.t.We denote such σ by σ_R.

Proof. Let $0 \le \Omega < b$. From Lemma 2.2 there exists $0 \le x < \alpha$ such that $\sigma(\,x) = \Omega$.

We consider $x_\Omega = \sup\{x\colon \sigma(x) = \Omega\}$. .Since$\lim_{x\to\alpha^-} \sigma(x) = b, x_\Omega < \alpha$

show that $\{x \in R\colon \sigma_R(x) > \Omega\} = R \cap (x_\Omega, \alpha)$.

Let $x \in R$ be such that $\sigma_R(x) > \Omega$. If $x \le x_\Omega$, from Lemma 2.2 $\sigma_R(x) \le \sigma_R(x_\Omega) = \Omega$ and this a contradiction. Thus $x \in R \cap (x_\Omega, \alpha)$.On the other hand,

if $x \in R \cap (x_\Omega, \alpha), \sigma_R(x) \ge \sigma_R(x_\Omega) = \Omega$ .If $\sigma_R(x) = \Omega$

thenx_Ω is not the supreme and this is another contradiction. Therefore$\sigma_R(x) > \Omega$.Then

$$\mu_{\sigma_R}(\Omega) = \mu\big(R \cap (x_\Omega, \alpha)\big). \tag{2.1}$$

Now, we consider $g(\,x) = x, 0 \le x < b$.

From (2.1) and the continuity of σ, we have $\mu_{\sigma_R}(\Omega) = b - \sigma(x_\Omega) = b - \Omega = \mu_g(\Omega)$.

In consequence σ_R and g are equimeasurable functions. If I is any μ-measurable subset of $[0,b)$, then $g^{-1}(I) = I$ is a μ-measurable set. From (Bennet and Sharpley 1988, Lemma 7.3), $\sigma_R^{-1}(I)$is μ-measurable and $\mu\big(\sigma_R^{-1}(I)\big) = \mu\big(g^{-1}(I)\big) = \mu(I)$. The proof is complete.

Let $f \in \Lambda_{\omega,\emptyset}$. By redefining f, if necessary, on a set of μ-measure zero , we may assume that $|f|$and f^* have the same non-null range, say $R(\,f)$. For $\Omega \in R(\,f)$, we consider $C_f(\Omega) \coloneqq \{0 \le x < \alpha\colon |f(x)| = \Omega\}$ and $I_f(\Omega) \coloneqq \left\{\epsilon > \frac{-1}{2}\colon f^*(1+2\epsilon) = \Omega\right\}$. So,

$\mu\left(C_f(\Omega)\right) = \mu\left(I_f(\Omega)\right) < \infty$. By Lemma 2.3, the function $\sigma_\Omega\colon C_f(\Omega) \to I_f(\Omega)$

defined by$\sigma_\Omega(x) = \mu_f(\Omega) + \mu\left(C_f(\Omega)\right) \cap [0,x]$is an m.p.t. Thus, the function

$$\sigma_f(x) = \sigma_\Omega(x)\left(x \in C_f(\Omega)\right) \tag{2.2}$$

is an m.p.t. from supp (f) onto supp(f^*).

Remark 2.4 Given $f \in \Lambda_{\omega,\emptyset}$ we can write

$$\sigma_f(x) = \mu_f(|f(x)|) + \mu(\{y: |f(y)| = |f(x)| \text{and } y \leq x\}).$$

If $\mu(\mathrm{supp}(f)) < \infty$, then σ_f is an m.p.t. from$[0,\alpha)$ into $[0,\alpha)$.

Lemma 2.5 Let $f, h \in \Lambda_{\omega,\emptyset}$and let$\Omega > 0$. If $\mu\left(C_f(\Omega)\right) = 0$, then μ_f is acontinuous function at Ω and $\lim_{1-2\epsilon\to 0} \mu_{f+h(1-2\epsilon)}(\Omega) = \mu_f(\Omega)$.

Proof. Since μ_f is a right continuous function, it is sufficient to show that
$\lim_{1-2\epsilon\to\Omega^-} \mu_f(1-2\epsilon) = \mu_f(\Omega)$. Let$((1-2\epsilon)_n)_n$be a sequence such that
$0 < (1-2\epsilon)_n \uparrow \Omega$ and$C_n = \{y: |f(y)| > (1-2\epsilon)_n\}$.

Clearly $C_{n+1} \subset C_n$and $\mu(C_1) < \infty$.As $\mu\left(C_f(\Omega)\right) = 0$, we get

$$\lim_{n\to\infty} \mu_f\left((1-2\epsilon)_n\right) = \mu\left(\bigcap_{n=1}^{\infty} C_n\right) = \mu(\{y: |f(y)| \geq \Omega\}) = \mu_f(\Omega).$$

Now we shall prove that$\lim_{1-2\epsilon\to 0}\mu_{f+h(1-2\epsilon)}(\Omega) = \mu_f(\Omega)$. Let $1-2\epsilon, 0 < |1-2\epsilon| < 1$.

Using properties of the distribution function we obtain

$$\mu_{f+h(1-2\epsilon)}(\Omega) = \mu_{f+h(1-2\epsilon)}\left(\left(1-\sqrt{|1-2\epsilon|}\right)\Omega + \sqrt{|1-2\epsilon|}\Omega\right)$$

$$\leq \mu_f\left(\left(1-\sqrt{|1-2\epsilon|}\right)\Omega\right) + \mu_h\left(\frac{\sqrt{|1-2\epsilon|}}{|1-2\epsilon|}\Omega\right) \tag{2.3}$$

and$\mu_f(\Omega) \leq \underline{\lim_{1-2\epsilon\to 0} \mu_{f+h(1-2\epsilon)}}(\Omega)$.

Since $h \in \Lambda_{\omega,\emptyset}$ we have$\lim_{1-2\epsilon\to 0} \mu_h\left(\frac{\sqrt{|1-2\epsilon|}}{|1-2\epsilon|}\Omega\right) = 0$.

In addition,$\lim_{1-2\epsilon\to 0} \mu_f\left(\left(1-\sqrt{|1-2\epsilon|}\right)\Omega\right) = \mu_f(\Omega)$.

So (2.3) implies that $\overline{\lim_{1-2\epsilon\to 0} \mu_{f+h(1-2\epsilon)}}(\Omega) \leq \mu_f(\Omega)$.

The proof is complete.

Remark 2.6 Let $f \in \Lambda_{\omega,\emptyset}$ and $\Omega > 0$. Clearly $\mu\left(\left\{y: \left|f(y)\chi_{\overline{C_f(\Omega)}}(y)\right| = \Omega\right\}\right) = 0$, where $\overline{A} = [0,\alpha) - A$. Thus, Lemma 2.5 implies that $\mu_{f\chi_{\overline{C_f(\Omega)}}}$ is continuous at Ω.

Lemma 2.7 Let $f, h \in \Lambda_{\omega,\emptyset}$ .If $\Omega > 0$ and $x \in C_f(\Omega)$,
then $\lim_{1-2\epsilon\to 0^+} \mu_{(f+(1-2\epsilon)h)\chi_{\overline{C_f(\Omega)}}}(|f(x) + (1-2\epsilon)h(x)|) = \mu_f(\Omega)$.

Proof. It is enough to operate on decreasing sequence $(1-2\epsilon)_n$, which we denote by$(1-2\epsilon)_n \downarrow 0$ .Since $\lim_{n\to\infty} |f(x)+(1-2\epsilon)_n h(x)| \frac{\sqrt{(1-2\epsilon)_n}}{(1-2\epsilon)_n} = \infty$ and $f, h \in \Lambda_{\omega,\emptyset}$, then

$$\lim_{n\to\infty} \mu_{h\chi_{\overline{C_f(\Omega)}}}\left(|f(x)+(1-2\epsilon)_n h(x)| \frac{\sqrt{(1-2\epsilon)_n}}{(1-2\epsilon)_n}\right) = 0\,. \tag{2.4}$$

In fact, (2.4) is obvious if $\alpha < \infty$. For $\alpha = \infty$, (2.4) follows from our assumption $\int_0^\infty w(1+2\epsilon) d\mu(1+2\epsilon) = \infty$. Using properties of the distribution function we obtain,

$$\begin{aligned}&\mu_{(f+(1-2\epsilon)_n h)\chi_{\overline{C_f(\Omega)}}}(|f(x)+(1-2\epsilon)_n h(x)|)\\ &\qquad \le \mu_{f\chi_{\overline{C_f(\Omega)}}}\left(|f(x)+(1-2\epsilon)_n h(x)|\left(1-\sqrt{(1-2\epsilon)_n}\right)\right)\\ &\qquad + \mu_{h\chi_{\overline{C_f(\Omega)}}}\left(|f(x)+(1-2\epsilon)_n h(x)| \frac{\sqrt{(1-2\epsilon)_n}}{(1-2\epsilon)_n}\right).\end{aligned}$$

So, (2.4) implies that

$$\overline{\lim_{n\to\infty}}\, \mu_{(f+(1-2\epsilon)_n h)\chi_{\overline{C_f(\Omega)}}}(|f(x)+(1-2\epsilon)_n h(x)|) \le \mu_f(\Omega). \tag{2.5}$$

On the other hand, as shown in (Bennet and Sharpley 1988), it is known that

$$\mu_f(\Omega) = \mu_{f\chi_{\overline{C_f(\Omega)}}}(\Omega) \le \lim_{n\to\infty} \mu_{(f+(1-2\epsilon)_n h)\chi_{\overline{C_f(\Omega)}}}(|f(x)+(1-2\epsilon)_n h(x)|). \tag{2.6}$$

From (2.5) and (2.6) the proof follows immediately.

Lemma 2.8 Let $f, h \in \Lambda_{\omega,\emptyset}$. If $\Omega \ge 0$ and $x \in C_f(\Omega)$,

Then

$$\lim_{1-2\epsilon\to 0^+} \mu\left(\left\{y \in \overline{C_f(\Omega)}: |f(y)+(1-2\epsilon)h(y)| = |f(x)+(1-2\epsilon)h(x)| \text{ and } y \le x\right\}\right) = 0.$$

Proof. Let$(1-2\epsilon)_n \downarrow 0, C_n :=$

$$\left\{y \in \overline{C_f(\Omega)}: |f(y)+(1-2\epsilon)_n h(y)| = |f(x)+(1-2\epsilon)_n h(x)| \text{ and } y \le x\right\}$$

and $D_n := \left\{y \in \overline{C_f(\Omega)} : \| f(y)| - f(x)|\| \le (1-2\epsilon)_n(|h(y)|+|h(x)|) \text{and } y \le x\right\}$.

Clearly $C_n \subset D_n \subset [0,x], D_{n+1} \subset D_n$ and $\bigcap_{n=1}^\infty D_n = \emptyset$.

Then $0 \le \lim_{n\to\infty} \mu(C_n) \le \lim_{n\to\infty} \mu(D_n) = \mu(\bigcap_{n=1}^\infty D_n) = 0$.

Lemma 2.9 Let $f, h \in \Lambda_{\omega,\emptyset}$. If$\Omega > 0$ and $x \in C_f(\Omega)$, then

$$\begin{aligned}&\lim_{1-2\epsilon\to 0^+} \mu(\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)h(y)| > |f(x)+(1-2\epsilon)h(x)|\})\\ &\qquad = \mu(\{y \in C_f(\Omega): (1-2\epsilon)g(f(y)h(y) > (1-2\epsilon)g(f(x)h(x)\}).\end{aligned}$$

Proof. Let $(1-2\epsilon)_n \downarrow 0$,

$$R_n := \left\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)_n h(y)| > |f(x)+(1-2\epsilon)_n h(x)| \text{ and } |h(y)| \le \frac{\Omega}{(1-2\epsilon)_n}\right\},$$

$(1-2\epsilon)_n := \left\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)_n h(y)| > |f(x)+(1-2\epsilon)_n h(x)| \text{ and} |h(y)| > \frac{\Omega}{(1-2\epsilon)_n}\right\}$ and

$R := \{y \in C_f(\Omega): (1-2\epsilon)g(f(y)h(y) > (1-2\epsilon)g(f(x)h(x)\}$.

As $\mu((1-2\epsilon)_n) \le \mu_h\left(\frac{\Omega}{(1-2\epsilon)_n}\right)$, $\lim_{n\to\infty} \mu((1-2\epsilon)_n) = 0$.

Then, it will be sufficient to prove that

$\lim_{n\to\infty} \mu(R_n) = \mu(R)$Let $N \in \mathbb{N}$ be such that if $n \ge \mathbb{N}$,

$$|f(x)+(1-2\epsilon)_n h(x)| = \Omega + (1-2\epsilon)_n (1-2\epsilon) g\big(f(x)\big)h(x).$$

Then, for $n \ge N, R_n \subset R$. In fact, if $y \in R_n$, $|h(y)| \le \frac{\Omega}{(1-2\epsilon)_n}$. Therefore,

$$|f(y)+(1-2\epsilon)_n h(y)| = \Omega + (1-2\epsilon)_n (1-2\epsilon) g\big(f(y)\big)h(y). \tag{2.7}$$

So,

$$\Omega + (1-2\epsilon)_n (1-2\epsilon) g\big(f(y)\big)h(y) = |f(y)+(1-2\epsilon)_n h(y)| > |f(x)+(1-2\epsilon)_n h(x)|$$
$$= \Omega + (1-2\epsilon)_n (1-2\epsilon) g\big(f(x)\big)h(x)$$

and consequently $(1-2\epsilon)g\big(f(y)\big)h(y) > (1-2\epsilon)g\big(f(x)\big)h(x)$.

For all $n \ge \mathbb{N}, R - R_n \subset \left\{y: |h(y)| > \frac{\Omega}{(1-2\epsilon)_n}\right\}$.

On the contrary, let $y \in R - R_n$ be with $|h(y)| \le \frac{\Omega}{(1-2\epsilon)_n}$. From (2.7) we have

$|f(y)+(1-2\epsilon)_n h(y)| = \Omega + (1-2\epsilon)_n (1-2\epsilon) g\big(f(y)\big)h(y) > \Omega + (1-2\epsilon)_n (1-2\epsilon) g\big(f(x)\big)h(x) = |f(x)+(1-2\epsilon)_n h(x)|$, which is a contradiction.

Since $\mu(R-R_n) \le \mu_h\left(\frac{\Omega}{(1-2\epsilon)_n}\right)$ we have

$$\lim_{n\to\infty} \mu(R_n) = \mu(R). \tag{2.8}$$

Lemma 2.10 Let $f, h \in \Lambda_{\omega,\emptyset}$. If $\Omega > 0$ and $x \in C_f(\Omega)$, then

$$\lim_{1-2\epsilon\to 0^+} \mu\big(\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)h(y)| = |f(x)+(1-2\epsilon)h(x)| \text{ and } y \le x\}\big)$$
$$= \mu\big(\{y \in C_f(\Omega): (1-2\epsilon)g\big(f(y)\big)h(y) = (1-2\epsilon)g\big(f(x)\big)h(x) \text{ and } y \le x\}\big).$$

Proof. Let$(1-2\epsilon)_n \downarrow 0$,

$$R_n \coloneqq \Big\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)_n h(y)| = |f(x)+(1-2\epsilon)_n h(x)|, |h(y)| \le \frac{\Omega}{(1-2\epsilon)_n} \text{ and } y \le x\Big\},$$

$$(1-2\epsilon)_n \coloneqq \Big\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)_n h(y)| = |f(x)+(1-2\epsilon)_n h(x)|, |h(y)| > \frac{\Omega}{(1-2\epsilon)_n} \text{ and } y \le x\Big\}$$

and $R \coloneqq \{y \in C_f(\Omega): (1-2\epsilon)g(f(y)h(y) = (1-2\epsilon)g(f(x)h(x) \text{ and } y \le x\}$.

Now,the proof follows in the same way as in Lemma 2.9

Theorem 2.11 Let $f, h \in \Lambda_{\omega,\emptyset}$ and let $\tau_{f,h}$be defined by (1.2).

(a) If $x \in \operatorname{supp}(f)$, $\lim_{1-2\epsilon\to 0^+} \sigma_{f+h(1-2\epsilon)}(x) = \tau_{f,h}(x)$;

(b)If $x \in \operatorname{supp}(h) - \operatorname{supp}(f)$, $\lim_{1-2\epsilon\to 0^+} \sigma_{f+h(1-2\epsilon)}(x) = \mu_f(0) + \mu_{h_{\operatorname{supp}(h)-\operatorname{supp}(f)}}(x)$.

Proof. (a) Let $x \in \operatorname{supp}(f)$and define $\Omega = |f(x)|$. We observe that for all sufficiently small $1-2\epsilon$, $x \in \operatorname{supp}\big(f+h(1-2\epsilon)\big)$. Hence, we get

$$\sigma_{(f+h(1-2\epsilon))}(x) = \mu_{(f+h(1-2\epsilon))\chi_{\overline{C_f(\Omega)}}}(|f(x)+(1-2\epsilon)h(x)|) + \mu\Big(\Big\{y \in \overline{C_f(\Omega)}: |f(y)+(1-2\epsilon)h(y)| = |f(x)+(1-2\epsilon)h(x)| \text{and } y \le x\Big\}\Big) + \mu\big(\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)h(y)| > |f(x)+(1-2\epsilon)h(x)|\}\big) + \mu\big(\{y \in C_f(\Omega): |f(y)+(1-2\epsilon)h(y)| = |f(x)+(1-2\epsilon)h(x)| \text{ and } y \le x\}\big).$$

Therefore, Lemmas 2.7 - 2.10 imply (a).

(b) Let $x \in \operatorname{supp}(h) - \operatorname{supp}(f)$. Suppose $\mu \ \operatorname{supp}(f) < \infty$.

For all $\epsilon > \frac{1}{2}$, $x \in \operatorname{supp}(f+(1-2\epsilon)h)$.Then ,we have

$$\begin{aligned}\sigma_{f+(1-2\epsilon)h}(x) &= \mu(\{y \in \operatorname{supp}(f): |f(y)+(1-2\epsilon)h(y)| > (1-2\epsilon)|h(x)|\}) \\ &\quad + \mu(\{y \in \operatorname{supp}(f): |f(y)+(1-2\epsilon)h(y)| = (1-2\epsilon)|h(x)| \text{and } y \le x\}) \\ &\quad + \sigma_{h_{\operatorname{supp}(h)-\operatorname{supp}(f)}}(x). \qquad (2.9)\end{aligned}$$

According to Lemma 2.8 the second term of (2.9) tends to zero. We only need to prove that the first term tends to$\mu_f(0)$.

$$R_{(1-2\epsilon)} \coloneqq \{y \in \operatorname{supp}(f): |f(y)+(1-2\epsilon)h(y)| > (1-2\epsilon)|h(x)|\},$$

$T_{(1-2\epsilon)} \coloneqq \{y \in \operatorname{supp}(f): |f(y)| \le (1-2\epsilon)(|h(y)|+|h(x)|)\}$.

Since$\lim_{1-2\epsilon\to 0^+} \mu(T_{1-2\epsilon}) = 0$ and

$$\text{supp}(f) - R_{1-2\epsilon} \subset T_{1-2\epsilon}, \tag{2.10}$$

then $\lim_{1-2\epsilon\to 0^+} \mu(R_{1-2\epsilon}) = \mu(\text{supp}(f)) = \mu_f(0)$.

Now, suppose that $\mu(\text{supp}(f)) = \infty$ .Given $M > 0$, we can choose $(1-2\epsilon)_1$ such that $\mu_f(|(1-2\epsilon)_1 h(x)|) > M$ and $\mu\left(C_f(|(1-2\epsilon)_1 h(x)|)\right) = 0$.

Then by Lemma 2.5 we obtain

$\lim_{1-2\epsilon\to 0} \mu_{f+(1-2\epsilon)h}(|(1-2\epsilon)_1 h(x)|) = \mu_f(|(1-2\epsilon)_1 h(x)|)M$. Thus, $\mu_{f+(1-2\epsilon)h}(|(1-2\epsilon)_1 h(x)|) > M$ for all sufficiently small $(1-2\epsilon)$.

It follows that $M < \mu_{f+h(1-2\epsilon)}(|(1-2\epsilon)h(x)|)$, for all sufficiently small $(1-2\epsilon)$.

Finally, as $\mu_{f+h(1-2\epsilon)}(|(1-2\epsilon)h(x)|) \leq \sigma_{f+h(1-2\epsilon)}(x)$, the proof of (b) is complete.

Definition 2.12 Let $f, h \in \Lambda_{\omega,\emptyset}$.We define

$$\rho_{f,h}(x) = \begin{cases} \tau_{f,h(x)} \text{if } x \in \text{supp}(f), \\ \mu_f(0) + \sigma_{h_{\text{supp}(h)-\text{supp}(f)}}(x) \text{ if } x \in \text{supp}(h) - \text{supp}(f) \end{cases}.$$

Example 2.13 Let $f, h \in \Lambda_{\omega,\emptyset}$.If $(\text{supp}(f)) < \infty$,

then $\rho_{f,h}$ is m.p.t. from $\text{supp}(f) \cup \text{supp}(h)$ onto $[0, \mu\,\text{supp}(f) \cup \text{supp}(h)))$ such that $|f| = f^* \circ \rho_{f,h}\mu$ -a.e. on $\text{supp}(f) \cup \text{supp}(h)$ .Let $f, h \in \Lambda_{\omega,\emptyset}$ and let $(1-2\epsilon)$ be non- zero real number. We denote

$F(1-2\epsilon) = \frac{\emptyset(|f+h(1-2\epsilon)| - \emptyset(|f|))}{1-2\epsilon}$.The next result is one of the main theorems of this section.

Theorem 2.14 Let $, h \in \Lambda_{\omega,\emptyset}$ .Then

$$\gamma^+_{\varphi_{\omega,\emptyset}}(f,h) = \int_{\text{supp}(f)} \omega(\rho_{f,h})\emptyset'(|f|)(1-2\epsilon)g(f)h\, d\mu + \emptyset'_+(0)\int_{\text{supp}(h)-\text{supp}(f)} \omega(\rho_{f,h})|h|\, d\mu\,, \tag{2.11}$$

where$\emptyset'_+(0)$is the right derivative of $\emptyset$ at 0. In (2.11) , we write $\omega(\infty) = 0$.

Proof. Assume that $\mu\,(\text{supp}(f)) < \infty$ .Let $\epsilon > \frac{1}{2}$. Clearly

$$\varphi_{\omega,\emptyset}(f + h(1-2\epsilon)) = \int_{\text{supp}(f+h(1-2\epsilon))} \omega(\sigma_{f+h(1-2\epsilon)})\emptyset\,(|f + h(1-2\epsilon)|)\, d\mu$$

and by Example 2.13 we have

$$\varphi_{\omega,\emptyset}(f) = \int_{\text{supp}(f)\,\cup\,\text{supp}(h)} \omega(\rho_{f,h})\,\emptyset(|f|)\, d\mu$$

As$\left(\omega\left(\rho_{f,h}\right)\right)^{*} = \omega$ in $[\,0, \mu(\operatorname{supp}(f)\ \cup\ \operatorname{supp}(h)))$, by the Hardy-Littlewood inequality (Bennet and Sharpley 1988)

$$\int_{\operatorname{supp}(f)\,\cup\,\operatorname{supp}(h)} \omega\left(\rho_{f,h}\right) \emptyset(|f + h(1-2\epsilon)|)\, d\mu$$
$$\leq \int_{0}^{\alpha} \omega(1+2\epsilon)\, \emptyset\left(\left(f + h(1-2\epsilon)\right)^{*}(1+2\epsilon)\right) d(1+2\epsilon).$$

In consequence , we get

$$\frac{1}{1-2\epsilon}\left(\int_{\operatorname{supp}(f+h(1-2\epsilon))} \omega\left(\sigma_{f+h(1-2\epsilon)}\right) \emptyset(|f + h(1-2\epsilon)|)\, d\mu\right.$$
$$\left. - \int_{\operatorname{supp}(f)\,\cup\,\operatorname{supp}(h)} \omega\left(\rho_{f,h}\right) \emptyset\,(|f + h(1-2\epsilon)|)\, d\mu\right) \geq 0.$$

Therefore

$$\frac{\varphi_{\omega,\emptyset}\left(f + h(1-2\epsilon)\right) - \varphi_{\omega,\emptyset}(f)}{1-2\epsilon}$$
$$\geq \int_{\operatorname{supp}(f)} \omega\left(\rho_{f,h}\right) F\,(1-2\epsilon)\, d\mu + P(1-2\epsilon), \tag{2.12}$$

where $P(1-2\epsilon) = \int_{\operatorname{supp}(h)-\,\operatorname{supp}(f)} \omega\left(\rho_{f,h}\right) \frac{\emptyset|h(1-2\epsilon)|}{1-2\epsilon}\, d\mu.$

Analogously with$\left(\,f + h(1-2\epsilon)\right)$ instead of f,we get the inequality

$$\frac{\varphi_{\omega,\emptyset}\left(f + h(1-2\epsilon)\right) - \varphi_{\omega,\emptyset}(f)}{1-2\epsilon}$$
$$\leq \int_{\operatorname{supp}(f+h(1-2\epsilon))\cap\operatorname{supp}(f)} \omega\left(\sigma_{f+h(1-2\epsilon)}\right) F\,(1-2\epsilon)\, d\mu + Q(1-2\epsilon), \tag{2.13}$$

where $Q(1-2\epsilon) = \int_{\operatorname{supp}(h)-\,\operatorname{supp}(f)} \omega\left(\sigma_{f+h(1-2\epsilon)}\right) \frac{\emptyset|h(1-2\epsilon)|}{1-2\epsilon}\, d\mu.$

Let $\frac{1}{2} < \epsilon\ < 0$. Since $\emptyset$ is a convex function, we have $\emptyset'(1+2\epsilon)1 + 2\epsilon \leq \emptyset\left(2(1+2\epsilon)\right)$ for all $\epsilon \geq \frac{-1}{2}$. In addition, the mean value Theorem implies that

$|F(1-2\epsilon)| \leq \emptyset'(\max\{|f + h(1-2\epsilon)|, |f|\})|h|.$

Therefore $\left|\omega\left(\left(\rho_{f,h}\right)\right) F(1-2\epsilon)\right| \leq \omega\left(\rho_{f,h}\right)\emptyset\left(2(|f| + |h|)\right).$

Also,for all sufficiently small $(1-2\epsilon), \epsilon > \frac{1}{2}$, from the proof of Theorem 2.11 we obtain $\rho_{f,h} \leq \sigma_{(f+h(1-2\epsilon))}$.So, $\left|\omega\left(\sigma_{f+h(1-2\epsilon)}\right) F(1-2\epsilon)\right| \leq \omega\left(\rho_{f,h}\right)\emptyset\left(2(|f|+|h|)\right)$ for all sufficiently small $(1-2\epsilon)$, $\epsilon > \frac{1}{2}$. Clearly,

$$\int_{\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right)\emptyset\left(2(|f|+|h|)\right) d\mu \leq \int_0^{\alpha} \omega\, \emptyset(2(|f|+|h|)^*) d(1+2\epsilon) < \infty,$$

then, from Theorem 2.11 and the Lebesgue Convergence Theorem we get

$$\lim_{1-2\epsilon \to 0^+} \int_{\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right) F(1-2\epsilon)\, d\mu = \int_{\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right) \emptyset'\,(|f|)(1-2\epsilon) g(f) h\, d\mu \quad (2.14)$$

and

$$\lim_{1-2\epsilon \to 0^+} \int_{\mathrm{supp}\left(f+h(1-2\epsilon)\right) \cap \mathrm{supp}(f)} \omega\left(\sigma_{f+h(1-2\epsilon)}\right) F(1-2\epsilon)\, d\mu$$

$$= \int_{\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right) \emptyset'\,(|f|)(1-2\epsilon) g(f) h\, d\mu. \quad (2.15)$$

If $(\mathrm{supp}(h) - \mathrm{supp}(f) = 0$, then $P(1-2\epsilon) \equiv 0$ and $Q(1-2\epsilon) \equiv 0$. So, (2.11) holds. Otherwise, for all sufficiently small $(1-2\epsilon)$, $\epsilon > \frac{1}{2}$,

$$\omega\left(\rho_{f,h}\right) \frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} \leq \omega\left(\sigma_{h_{\mathrm{supp}(h)-\mathrm{supp}(f)}}\right) \emptyset(|h|) \text{on}\, \mathrm{supp}(h) - \mathrm{supp}(f).$$

Since

$$\int_{\mathrm{supp}(h)-\mathrm{supp}(f)} \omega\left(\sigma_{h_{\mathrm{supp}(h)-\mathrm{supp}(f)}}\right) \emptyset(|h|) \leq \varphi_{\omega,\emptyset}(h) < \infty, \quad (2.16)$$

the Lebesgue Convergence Theorem implies that

$$\lim_{1-2\epsilon \to 0^+} P(1-2\epsilon) = \emptyset'_+\,(0) \int_{\mathrm{supp}(h)-\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right)|h| d\mu. \quad (2.17)$$

On the other hand ,

$$\lim_{1-2\epsilon \to 0^+} \omega\left(\sigma_{f+h(1-2\epsilon)}\right) \frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} = \emptyset'_+(0)\, \omega\left(\rho_{f,h}\right)|h| \text{on}\, \mathrm{supp}(h) - \mathrm{supp}(f)$$

and for all sufficiently small $(1-2\epsilon)$, $\epsilon > \frac{1}{2}$, (2.9) implies

$$\omega\left(\sigma_{f+h(1-2\epsilon)}\right) \frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} \leq \omega\left(\sigma_{h_{\mathrm{supp}(h)-\mathrm{supp}(f)}}\right) \emptyset(|h|) \text{on } \mathrm{supp}(h) - \mathrm{supp}(f). \quad (2.18)$$

According to (2.16) and the Lebesgue Convergence Theorem ,

$$\lim_{1-2\epsilon \to 0^+} Q(1-2\epsilon) = \emptyset'_+\,(0) \int_{\mathrm{supp}(h)-\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right)|h| d\mu. \quad (2.19)$$

Therefore, (2.14), (2.15), (2.17) and (2.19) imply (2.11) .

Now assume that $\mu\ (\mathrm{supp}(f))\ =\ \infty$ .Similarly, to the proof of (2.12) and(2.13) ,we can obtain

$$\int_{\mathrm{supp}(f)} \omega\big(\rho_{f,h}\big)F(1-2\epsilon)d\mu \le \frac{\varphi_{\omega,\emptyset}\big(f+h(1-2\epsilon)\big)-\varphi_{\omega,\emptyset}(f)}{1-2\epsilon} \le$$

$$\int_{\mathrm{supp}(f+h(1-2\epsilon))\cap\mathrm{supp}(f)} \omega\big(\rho_{f,h}\big)F(1-2\epsilon)d\mu + Q(1-2\epsilon). \qquad (2.20)$$

Proceeding as before, we get (2.14) and (2.15).

If $\mu(\mathrm{supp}(h)-\ \mathrm{supp}(f)\ =0$, $Q(1-2\epsilon)\equiv\ 0$ and (2.11) is true.

In opposite case,

$\lim_{1-2\epsilon\to 0^+} w\big(\sigma_{f+h(1-2\epsilon)}\big)\frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon}=0.$

From (2.16),(2.18) and the Lebesgue Convergence Theorem,

$\lim_{1-2\epsilon\to 0^+} Q(1-2\epsilon)=0.$

The proof is complete. In the next theorem, we obtain the one-sided Gateaux derivative of the Lebesgue norm in terms of the one-sided Gateaux derivative of the functional $\varphi_{\omega,\emptyset}$.

Theorem 2.15 Let $f,h\in\ \Lambda_{\omega,\emptyset}$, $f\ \neq 0$ .Then

$$\gamma^{+}_{\|\cdot\|_{\omega,\emptyset}}(f,h)=\frac{\gamma^{+}_{\varphi_{\omega,\emptyset}}\left(\frac{f}{\|f\|_{\omega,\emptyset}},h\right)}{\gamma^{+}_{\varphi_{\omega,\emptyset}}\left(\frac{f}{\|f\|_{\omega,\emptyset}},\frac{f}{\|f\|_{\omega,\emptyset}}\right)}. \qquad (2.21)$$

Proof. If $h\ =0$, (2.21) is obvious. Now suppose that $h\neq\ 0$.

For all $(1-2\epsilon)$, $0<\ 1-2\epsilon<\frac{\|f\|_{\omega,\emptyset}}{2\|h\|_{\omega,\emptyset}}$, we denote

$K(1-2\epsilon)=\frac{\emptyset\left(\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right)-\emptyset\left(\frac{|f|}{\|f\|_{\omega,\emptyset}}\right)}{1-2\epsilon}$ and $G(1-2\epsilon)=\varphi_{\omega,\emptyset}\left(\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right).$

First, we assume that $\mu(\mathrm{supp}(f))<\infty$ and we consider

$P(1-2\epsilon)=\int_{\mathrm{supp}(h)-\mathrm{supp}(f)}\omega\big(\rho_{f,h}\big)\frac{\emptyset\left(\frac{|h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right)}{1-2\epsilon}d\mu$and

$$Q(1-2\epsilon)=\int_{\mathrm{supp}(h)-\mathrm{supp}(f)}\omega\big(\sigma_{f+h(1-2\epsilon)}\big)\frac{\emptyset\left(\frac{|h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right)}{1-2\epsilon}d\mu.$$

Proceeding analogously to the proof of Theorem 2.14, we can obtain

$$\int_{\mathrm{supp}(f)}\omega\ \big(\rho_{f,h}\big)K(1-2\epsilon)d\mu+P(1-2\epsilon)\le\frac{G(1-2\epsilon)-G(0)}{1-2\epsilon}\le$$

$$\int_{\operatorname{supp}(f+h(1-2\epsilon))\cap\operatorname{supp}(f)} \omega\left(\sigma_{f+h(1-2\epsilon)}\right)K(1-2\epsilon)d\mu+Q(1-2\epsilon), \tag{2.22}$$

Let $0<1-2\epsilon\leq\min\left\{1,\frac{\|f\|_{\omega,\emptyset}}{2\|h\|_{\omega,\emptyset}}\right\}$.

Adding and subtracting $\frac{|f|\|f\|_{\omega,\emptyset}}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}\|f\|_{\omega,\emptyset}}$ to the expression $\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|_{\omega,\emptyset}}$ and applying the triangular inequality we obtain

$$\left|\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|_{\omega,\emptyset}}\right|\leq\frac{2(1-2\epsilon)M}{\|f\|_{\omega,\emptyset}}(|f|+|h|),$$

where $M=1+\frac{\|h\|_{\omega,\emptyset}}{\|f\|_{\omega,\emptyset}}$.In consequence, the Main Value Theorem implies that

$$|K(1-2\epsilon)|\leq\frac{\emptyset'\left(\max\left\{\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}},\frac{|f|}{\|f\|_{\omega,\emptyset}}\right\}\right)}{1-2\epsilon}\left|\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|_{\omega,\emptyset}}\right|$$

$$\leq\emptyset'\left(\frac{2M}{\|f\|_{\omega,\emptyset}}(|f|+|h|)\right)\frac{2M}{\|f\|_{\omega,\emptyset}}(|f|+|h|)\leq\emptyset\left(\frac{4M}{\|f\|_{\omega,\emptyset}}(|f|+|h|)\right).$$

From the Lebesgue Convergence Theorem, we can show that

$$\lim_{1-2\epsilon\to0^+}\int_{\operatorname{supp}(f)}\omega\left(\rho_{f,h}\right)K(1-2\epsilon)d\mu$$

$$=\lim_{1-2\epsilon\to0^+}\int_{\operatorname{supp}(f+h(1-2\epsilon))\cap\operatorname{supp}(f)}\omega\left(\sigma_{f+h(1-2\epsilon)}\right)K(1-2\epsilon)d\mu$$

$$=\int_{\operatorname{supp}(f)}\omega\left(\rho_{f,h}\right)\emptyset'\left(\frac{|f|}{\|f\|_{\omega,\emptyset}}\right)\left(\frac{(1-2\epsilon)g(f)h}{\|f\|_{\omega,\emptyset}}\right.$$

$$\left.-\frac{|f|}{\|f\|_{\omega,\emptyset}^2}\gamma^+_{\|\cdot\|_{\omega,\emptyset}}(f,h)\right)d\mu \tag{2.23}$$

and

$$\lim_{1-2\epsilon\to0^+}P(1-2\epsilon)=\lim_{1-2\epsilon\to0^+}Q(1-2\epsilon)=\emptyset'_+(0)\int_{\operatorname{supp}(h)-\operatorname{supp}(f)}\omega\left(\rho_{f,h}\right)\frac{|h|}{\|f\|_{\omega,\emptyset}}d\mu. \tag{2.24}$$

Thus, from (2.22) -(2.24), we have

$$\lim_{1-2\epsilon\to 0^+}\frac{G(1-2\epsilon)-G(0)}{1-2\epsilon}$$

$$=\emptyset'_+(0)\int_{\mathrm{supp}(h)-\mathrm{supp}(f)}\omega(\rho_{f,h})\frac{|h|}{\|f\|_{\omega,\emptyset}}d\mu$$

$$+\int_{\mathrm{supp}(f)}\omega\left(\rho_{f,h}\right)\emptyset'\left(\frac{|f|}{\|f\|_{\omega,\emptyset}}\right)\left(\frac{(1-2\epsilon)g(f)h}{\|f\|_{\omega,\emptyset}}\right.$$

$$\left.-\frac{|f|}{\|f\|^2_{\omega,\emptyset}}\gamma^+_{\|\cdot\|_{\omega,\emptyset}}(f,h)\right)d\mu. \tag{2.25}$$

Since $\varphi_{\omega,\emptyset}\left(\frac{g}{\|g\|_{\omega,\emptyset}}\right)=1$ for any $g\in\Lambda_{\omega,\emptyset}-\{0\}$(see Kamińska 1990) we have

$$G(1-2\epsilon)=1$$

for $0\le 1-2\epsilon<\frac{\|f\|_{\omega,\emptyset}}{2\|h\|_{\omega,\emptyset}}$.Therefore, from (2.25) and Theorem 2.14, we get (2.21) .

The case $\mu(\mathrm{supp}(f))=\infty$ follows in a similar way without using $P(1-2\epsilon)$ and after proving that

$\lim_{1-2\epsilon\to 0^+}Q(1-2\epsilon)=0.$

3. characterization of smooth points for the Luxemburg norm

We let X be a Banach space and let $T: X\to R^+$ be a convex functional. The following example shows that the set of smooth points of the functional T, in general, is not equal to the set of smooth points of the Minkowski functional of$\{f\in X: T(f)\le 1\}$(Levis and Cuenya, 2007).

Example 3.1 In a Hilbert space X define the continuous convex function

$$T(f)=\begin{cases}\|f\| & \text{if} \quad \|f\|>1,\\ 1 & \text{if} \quad \|f\|\le 1.\end{cases}$$

It is not Gateaux differentiable at any point f of norm |, but the Minkowski functional of $\{f\in X: T(f)\le 1\}$. (which is the closed unit ball) is just thenorm, which is infinitely Gateaux differentiable everywhere except at the origin. We consider the sets

$\mathcal{E}^{\omega,\emptyset}:=\left\{f\in\Lambda_{\omega,\emptyset}-\{0\}: \mu\{|f|=1-2\epsilon\}=0 \text{ for any } \epsilon>\frac{1}{2}\right\}$and

$$\Delta^{\omega,\emptyset}:=\mathcal{E}^{\omega,\emptyset}\cap\left\{f\in\Lambda_{\omega,\emptyset}:\mu\{f=0\}=0 \text{ on } \mu_f(0)=\infty\right\}.$$

In (Levis and Cuenya 2004), we have proved that $f\in\Lambda_{\omega,\emptyset}$is asmooth point of $\varphi_{\omega,\emptyset}$ if $f\in\mathcal{E}^{\omega,\emptyset}\left(f\in\Delta^{\omega,\emptyset}\right)$when $\emptyset'_+(0)=0(\emptyset'_+(0)>0)$.

It is well-known that if X is a Banach space and $T: X \to R$ is a convex functional then for all $f, h \in X$, $\gamma_T^+(f,h)$and$\gamma_T^-(f,h)$always exist and the equality $\gamma_T^+(f,h) = -\gamma_T^-(f,-h)$ holds (Pinkus, 1989), we showed a relation between the one-sided Gateaux derivative for the functional $\varphi_{\omega,\emptyset}$ and the one-sided Gateaux derivative for the Luxemburg norm. Consequently, f is a smooth point of the Luxemburg norm if and only if $\frac{f}{\|f\|_{\omega,\emptyset}}$is a smooth point for $\varphi_{\omega,\emptyset}$ .The next theorm follows immediately.

Theorem 3.2 The set of smooth points for the Luxemburg norm is $\mathcal{E}^{\omega,\emptyset}(\Delta^{\omega,\emptyset})if\emptyset'_+(0) = 0(\emptyset'_+(0) > 0)$.

Remark 3.3 It is well-known that $\mathcal{E}^{\omega,\emptyset}$ and $\Delta^{\omega,\emptyset}$ are dense sets in the $\Lambda_{\omega,\emptyset}$ because the points of Gateaux- differentiability of the norm in a separable space always from a dense set (Phelps, 1989).

4.Characterization of best approximants

We characterize the set of best approximants from convex closed sets using the one-sided Gateaux derivative. Moreover, we establish a relation between the best $\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set. Let $f, h \in \Lambda_{\omega,\emptyset}$.We denote by

$$\mathcal{A}_f := \{\sigma: \mathrm{supp}(f) \to \mathrm{supp}(f^*) : \sigma \text{ is m.p.t . and } |f| = f^* \circ \sigma, \mu - \text{a.e. on } \mathrm{supp}(f)\}$$

and

$$T_{f,h} = -\emptyset'_+(0) \int_{\mathrm{supp}(h)-\mathrm{supp}(f)} \omega\left(\rho_{f,h}\right)|h| d\mu. \quad (4.1)$$

In (4.1), we write $\omega(\infty) = 0$.

Theorem 4.1 Let $K \subset \Lambda_{\omega,\emptyset}$ be a convex closed set, let $f, h \in \Lambda_{\omega,\emptyset} - K$ and let $h^\bullet \in K$. Then the following statements are equivalent:

(a) $h^\bullet \in P_{\varphi_{\omega,\emptyset}}(f,K)$;

(b) $\int_{\mathrm{supp}(f-h^\bullet)} \omega\left(\rho_{f-h^\bullet,h^\bullet-h}\right)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g\,(f-h^\bullet)(h^\bullet-h)d\mu \geq T_{f-h^\bullet,h^\bullet-h}$for all $h \in K$;

(c) $\sup_{\sigma\in\mathcal{A}_{f-h^\bullet}} \int_{\mathrm{supp}(f-h^\bullet)} \omega\,(\sigma)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu \geq T_{f-h^\bullet,h^\bullet-h}$ for all $h \in K$.

In addition, if $T_{f-h^\bullet,h^\bullet-h=0}$,these statements imply

(d)$\varphi_{\omega,\emptyset}(f-h^\bullet) \leq \int_0^\alpha \omega(1+2\epsilon)\emptyset'\left((f-h^\bullet)^*(1+2\epsilon)\right)(f-h)^*(1+2\epsilon)d(1+2\epsilon)$

for all $h \in K$.

Proof. The implication (b) ⇒ (c) is obvious.

(a)⇔(b). This is an immediate consequence of Theorem 2.14 and (Pinkus 1989, Theorem 1.6), because this theorem still holds if we replace the norm$\|\cdot\|$ by the functional$\varphi_{\omega,\emptyset}$.

(c)⇒(b). Let $h \in K$ and $\sigma \in \mathcal{A}_{f-h^\bullet}$ .Taking σ , $f - h^\bullet$ and $h^\bullet - h$ instead of $\rho_{f,h}, f$ and h respectively in (2.12) or (2.20) we have

$$\int_{\mathrm{supp}(f-h^\bullet)} \omega(\sigma)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu - T_{f-h^\bullet,h^\bullet-h} \leq \Upsilon^+_{\varphi_{\omega,\emptyset}}(f-h^\bullet)(h^\bullet-h).$$

By hypothesis and Theorem 2.14, we get

$$0 \leq \sup_{\sigma\in\mathcal{A}_{f-h^\bullet}} \int_{\mathrm{supp}(f-h^\bullet)} \omega(\sigma)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu - T_{f-h^\bullet,h^\bullet-h}$$
$$\leq \Upsilon^+_{\varphi_{\omega,\emptyset}}(f-h^\bullet, h^\bullet-h)$$
$$= \int_{\mathrm{supp}(f-h^\bullet)} \omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu - T_{f-h^\bullet,h^\bullet-h}.$$

(b)⇒(d) .Assume$T_{f-h^\bullet,h^\bullet-h} = 0$ and let $h \in K$ .Since for all $\epsilon > \frac{-1}{2}$,

$\emptyset(1+2\epsilon) \leq \emptyset'(1+2\epsilon)1+2\epsilon$, then by hypothesis and the Hardy-Littlewood inequality we have

$$\varphi_{\omega,\emptyset}(f-h^\bullet) \leq \int_{\mathrm{supp}(f-h^\bullet)} \omega\left(\rho_{f-h^\bullet,h^\bullet-h}\right)\emptyset'(|f-h^\bullet|)|f-h^\bullet|d\mu$$
$$\leq \int_{\mathrm{supp}(f-h^\bullet)} \omega\left(\rho_{f-h^\bullet,h^\bullet-h}\right)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(f-h)d\mu.$$

According to (Bennet and Sharpley 1988, Proposition7.2)

$$\omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|) \sim \omega\emptyset'((f-h^\bullet)^*)$$

and $\omega\emptyset'((f-h^\bullet)^*)$is anon-increasing function. So,

$$\varphi_{\omega,\emptyset}(f-h^\bullet) \leq \int_0^\propto \omega\emptyset'\left((f-h^\bullet)^*(1+2\epsilon)\right)(f-h)^*(1+2\epsilon)d(1+2\epsilon).$$

The following example shows, that the implication (d)⇒(a) of Theorem 4.1 is not true in general.

Example 4.2 Let $\alpha = 1$. We consider

$$\phi(1+2\epsilon)=\begin{cases} e^{(1+2\epsilon)}-2(1+\epsilon)\text{if } \dfrac{-1}{2}\leq\epsilon\leq 0,\\ (e-2)(1+2\epsilon)^{\frac{e-1}{e-2}} \text{ if } \epsilon>0.\end{cases}$$

$K\coloneqq\{h\in\Lambda_{\omega,\phi}: h \text{ is constant}\}$and $f=\chi_{[0,\frac{1}{2})}$.

It is easy to see that

$\varphi_{\omega,\phi}(f)=(e-2)\int_0^{\frac{1}{2}}\omega(1+2\epsilon)d(1+2\epsilon)$.

On the other hand, for $h\in K$ and $\frac{-1}{2}\geq\ \epsilon>\frac{-1}{4}$,

we have$(f-h)^*(1+2\epsilon)\geq\frac{1}{2}$.Thus, for all $h\in K$,

$\int_0^1\omega(1+2\epsilon)\phi'\left(f^*(1+2\epsilon)\right)(f-h)^*(1+2\epsilon)d(1+2\epsilon)\geq\frac{e-1}{2}\int_0^{\frac{1}{2}}\omega(1+2\epsilon)d(1+2\epsilon)$.Consequently, (d) is true for$h^\bullet=0$. However, $P_{\varphi_{\omega,\phi}}(f,K)=\{\frac{1}{2}\}$.

Nevertheless, we show in the next theorem that (d)⇒(a) holds when
$\phi(1+2\epsilon)=(1+2\epsilon)^{1+\epsilon}$.

Theorem 4.3 Let $K\subset L_{(\omega,1+\epsilon)}$ be a convex closed set, let $0\leq\epsilon<\infty$,
let $f\in L_{(\omega,1+\epsilon)}-K$ and let $h^\bullet\in K$.Then, the following statements are equivalent:

(a) $h^\bullet\in P_{\|\cdot\|_{(\omega,1+\epsilon)}}(f,k)$.

(b) $\|f-h^\bullet\|_{(\omega,1+\epsilon)}^{(1+\epsilon)}\leq\int_0^\alpha\omega(1+2\epsilon)\left((f-h^\bullet)^*(1+2\epsilon)\right)^\epsilon(f-h)^*(1+2\epsilon)d(1+2\epsilon)$for all $h\in K$.

Proof. If $\epsilon=0$, this is obvious. Assume that $0<\epsilon<\infty$. (a)⇒(b) is an immediate consequence of Theorem 4.1.

(b)→(a) .Let $(1+\epsilon)$ and $(1-\epsilon)$ be conjugate numbers and let $h\in K$.From hypothesis and Hölder inequality, we get

$$\|f-h^\bullet\|_{(\omega,1+\epsilon)}^{(1+\epsilon)}\leq\int_0^\alpha\omega(1+2\epsilon)\left((f-h^\bullet)^*(1+2\epsilon)\right)^\epsilon(f-h)^*(1+2\epsilon)d(1+2\epsilon)$$
$$=\int_0^\alpha\omega(1+2\epsilon)^{\frac{1}{1-\epsilon}}\left((f-h^\bullet)^*(1+2\epsilon)\right)^\epsilon\omega(1+2\epsilon)^{\frac{1}{1+\epsilon}}(f-h)^*(1+2\epsilon)d(1+2\epsilon)\leq\|f-h^\bullet\|_{(\omega,1+\epsilon)}^\epsilon\|f-h\|_{(\omega,1+\epsilon).}$$

So, the proof is complete. Next, we establish a relation between the best $\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set K.

Theorem 4.4 Let $f,\in\ \Lambda_{\omega,\emptyset}$ and $K\subset\ \Lambda_{\omega,\emptyset}$ be convex set such that

$\delta = E_{\|\cdot\|_{\omega,\emptyset}}(f,K) > 0$.Then,$h^{\bullet} \in P_{\|\cdot\|_{\omega,\emptyset}}(f,K)$if and only if $\frac{h^{\bullet}}{\delta} \in P_{\varphi_{\omega,\emptyset}}\left(\frac{f}{\delta},\frac{K}{\delta}\right)$.

Proof. It follows immediately from (Pinkus1989, Theorem 1.6 and Theorem 2.15).

Remark 4.5 Theorem 4.4 is known for arbitrary sets K in modular space (see Kilmer and Kozlowski 1990).

Henceforth, we consider $\alpha < \infty, K \coloneqq \{g \in \Lambda_{\omega,\emptyset}: g \text{ is constant}\}$and

$f \in \Lambda_{\omega,\emptyset}$.Clearly,$P_{\varphi_{\omega,\emptyset}}(f,k)$is a nonempty and compact interval. We denote

$\underline{f} = \min P_{\varphi_{\omega,\emptyset}}(f,k)and\overline{f} = \max P_{\varphi_{\omega,\emptyset}}(f,k)$.

As a direct consequence of (Pinkus,1989, Theorem 1.6) we have that $c \in P_{\varphi_{\omega,\emptyset}}(f,k)$ if and only if

$$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0 \text{ and } \gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq 0. \tag{4.2}$$

The next characterization of best constant $\varphi_{\omega,\emptyset}$-approximants of f follows from (4.2) and Theorem 2.14.

Theorem 4.6 Let $f, \in \Lambda_{\omega,\emptyset}$. Then $c \in P_{\varphi_{\omega,\emptyset}}(f,k)$ if and only if the following statements hold:

(a)$\int_{f\geq c} \omega\left(\rho_{f-c,1}\right)\emptyset'(f-c)d\mu \geq \int_{f<c} \omega\left(\rho_{f-c,1}\right)\emptyset'(c-f)d\mu$

and

(b)$\int_{f\leq c} \omega\left(\rho_{c-f,1}\right)\emptyset'(c-f)d\mu \geq \int_{f>c} \omega\left(\rho_{c-f,1}\right)\emptyset'(f-c)d\mu$.

We write, $\emptyset'(0) \coloneqq \emptyset'_{+}(0)$ .According to Theorem 4.4 and 4.6, we obtain the following characterization of best constant approximants:

Corollary 4.7 Let $f \in \Lambda_{(\omega,1+\epsilon)} - K$.Then $c \in P_{\|\cdot\|_{\omega,\emptyset}}(f,k)$if and only if the following statements hold:

(a)$\int_{f\geq c} \omega\left(\rho_{f-c,1}\right)\emptyset'\left(\frac{f-c}{\|f-c\|_{\omega,\emptyset}}\right)d\mu \geq \int_{f<c} \omega\left(\rho_{f-c,1}\right)\emptyset'\left(\frac{c-f}{\|c-f\|_{\omega,\emptyset}}\right)d\mu$ and

(b)$\int_{f\leq c} \omega\left(\rho_{c-f,1}\right)\emptyset'\left(\frac{c-f}{\|c-f\|_{\omega,\emptyset}}\right)d\mu \geq \int_{f>c} \omega\left(\rho_{c-f,1}\right)\emptyset'\left(\frac{f-c}{\|f-c\|_{\omega,\emptyset}}\right)d\mu$.

Now our purpose is to give away to construct the best $\varphi_{\omega,\emptyset}$-approximants $\underline{f}$ and $\overline{f}$.

We begin with three lemmas.

Lemma 4.8 If $c < d$ then for all $0 \leq x \leq \alpha$ we have

(a)$\mu_{f-c}(f(x)-c) \leq \mu_{f-d}(f(x)-d)$ if $f(x) \geq d$

and

(b)$\mu_{f-d}\big(d-f(x)\big) \le \mu_{f-c}\big(c-f(x)\big)$if $f(x) < c$.

Proof. (a) Suppose $f(x) \ \ge d$, clearly,

$$\mu(\{y: 2d - f(x) \le f(y) \le f(x)\}) \le \mu(\{y: 2c - f(x) \le f(y) \le f(x)\}).$$

Therefore,

$$\mu(\{y: |f(y) - d| \le f(x) - d\}) \le \mu(\{y: |f(y) - c| \le f(x) - c\})$$

and consequently (a) holds.

(b). Now suppose that $(x) < c$. Clearly

$$\mu(\{y: |f(y) - c| \le c - f(x)\}) \le \mu(\{y: |f(y) - d| \le d - f(x)\})$$

and thus (b) is true.

Lemma 4.9 If $c < d$ then for $0 \le x \le \alpha$ we have:

(a) $\rho_{f-c,1}(x) \le \rho_{f-d,1}(x)$ if $f(x) \ge d$ and

(b)$\rho_{f-d,1}(x) \le \rho_{f-c,1}(x)$ if $f(x) < c$.

Proof. (a). Suppose $f(\,x) > \ d$. Since $f(\,x) > c$, from Lemma 4.8 we get

$$\rho_{f-d,1}(x) = \mu_{f-d}(f(x) - d) + \mu(\{y: f(y) = f(x) \text{and } y \le x\})$$
$$\ge \mu_{f-c}(f(x) - c) + \mu(\{y: f(y) = f(x) \text{and } y \le x\}) = \rho_{f-c,1}(x).$$

Now suppose that $f(x) = d$.

As

$$\mu_{f-c}(d-c) \le \mu_{f-d}(0) \text{ and } x \in \operatorname{supp}(f-c)\text{, then } \rho_{f-c,1}(x)$$
$$= \mu_{f-c}(d-c) + \mu(\{y: f(y) = d \text{ and } y \le x\})$$
$$\le \mu_{f-d}(0) + \mu(\{y: f(y) = d \text{ and } y \le x\}) = \rho_{f-d,1}(x).$$

(b). Assume $f(\,x) \ < c$. Since

$$\mu_{f-d}\big(d - f(x)\big) + \mu(\{y: f(y) - d = d - f(x)\}) \le \mu_{f-c}\big(c - f(x)\big),$$

we have$\rho_{f-d,1}(x) \le \mu_{f-c}\big(c - f(x)\big) + \mu(\{y: f(y) = f(x) \text{ and } y \le x\,\}) \le \rho_{f-c,1}(x)$.

Lemma 4.10 Let $f \in \Lambda_{(\omega,1+\epsilon)}$. If $c \le d$, then

(a)$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f-d,1) \le \gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1)$and

(b) $\gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \le \gamma^{+}_{\varphi_{\omega,\emptyset}}(d-f,1)$.

Proof. (a) We will show that

$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f-d,1) \le \gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1)$if $c < d$.

We define

$$P(u) = \int_{f \geq u} \omega\left(\rho_{f-u,1}\right)\emptyset'(f-u)d\mu \text{ and } Q(u) = \int_{f<u} \omega\left(\rho_{f-u,1}\right)\emptyset'(u-f)d\mu.$$

Clearly

$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f-u,1) = P(u) - Q(u).$

It will be sufficient to prove that P is a non-increasing function and Q is a non-decreasing function.

Since ω is non-increasing $\emptyset'_{+}$is a non-decreasing and

$$\{y: f(y) \geq d\} \subset \{y: f(y) > c\},$$

then from Lemma 4.9 (a) we have

$$P(d) \leq \int_{f>c} \omega\left(\rho_{f-c,1}\right) \emptyset'(f-c)d\mu \leq P(c).$$

Similarly, from Lemma 4.9 (b),

$$Q(c) \leq \int_{f<d} \omega\left(\rho_{f-d,1}\right)\emptyset'(d-f)d\mu = Q(d).$$

(b). Replacing in (a),f, c and d by $-f$, $-d$ and$-c$ respectively, we obtain (b).

Theorem 4.11 Let $f \in \Lambda_{(\omega,1+\epsilon)}$.Then

$\overline{f} = \max\left\{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0\right\}and\underline{f} = \min\left\{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq 0\right\}$.

Proof. Suppose that there exists $c, c > \overline{f}$, such that

$$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0. \tag{4.3}$$

By Lemma 4.10,

$$\gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq \gamma^{+}_{\varphi_{\omega,\emptyset}}\left(\overline{f}-f,1\right) \geq 0. \tag{4.4}$$

Then, (4.2)-(4.4) imply that $c \in P_{\varphi_{\omega,\emptyset}}(f,k)$, a contradiction.Thus,

$\overline{f} = \max\left\{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0\right\}$.

Similarly, we can see that$\underline{f} = \min\left\{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq 0\right\}$ (Levis and Cuenya 2007).

Conclusion

By Letting $\Lambda_{\omega,\emptyset}$ be the Orlicz-Lorentz space. We study Gateaux differentiability of the functional $\varphi_{\omega,\emptyset}(f) = \int_0^\infty \emptyset(f^*)\omega$ and of the Luxemburg norm. More precisely, we obtain the one-sided Gateaux derivatives in both cases and we characterize those points where the Gateaux derivative of norm exists. We give a characterization of best $\varphi_{\omega,\emptyset}$-approximants from convex closed subsets and we establish a relation between best$\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set. A characterization of best constant

$\varphi_{\omega,\emptyset}$-approximants and the algorithm to construct the best constant for maximum and minimum $\varphi_{\omega,\emptyset}$-approximants are given.

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Divisibility Test of Numbers by Some Prime Ones (7, 13, 17, 19, 23, 29)

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Abstract

The purpose of this paper is finding rules to know whether any number is divisible by some primes (7, 13, 17, 19, 23, 29), so that we use divisibility test and concept of congruence relation in numbers theory. The conclusion of the study arrived that we can know that any natural number is divisible by any prime without doing division.

Keywords: *Divisibility Test, Prime, Congruence.*

اختبارقابلية قسمة الأعدادعلى بعض الأعداد الأولية [7, 13, 17, 19, 23, 29]

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مُسْتَخْلَص

الغرض من هذه الورقة هو إيجاد قواعد لمعرفة ما إذا كان أيَّ عدد يقبل القسمة على بعض الأعداد الأولية [7, 13, 17, 19, 23, 29]ولذلك استخدمنا قابلية القسمة ومفهوم التطابق في نظرية العدد. وخلصنا إلى أنه يمكن معرفة قابلية القسمة لأيَّ عدد طبيعي على عدد أولي بدون إجراء عملية القسمة.

كلمات مفتاحية: *اختبارقابلية القسمة، الأعداد الأولية، التطابق.*

Introduction

Number theory contains the concept of congruence which considered other expression of division. Divisibility test of numbers by some prime (7, 13, 17, 19, 23, 29) has rules which used to check whether any positive integer is divided by these prime numbers. Applying the methods adding or subtracting, the case may be from the remaining truncated number is divisible by the given number. While carrying the process either during addition or subtraction any time the last digit is zero, which has to be ignored. Addition or subtraction check out the divisibility rule of some other numbers.

Basic concepts

Divisibility Test

Any integer a is said to be divisible by any integer b if there exist an integer c such that a = b c ($b \neq 0$) in symbols we wrote $b \mid a$, read (b divided a) or a is divisible by b, if b does not divide a we wrote write $b \nmid a$.

Theorem (2-1)

If a, b, c, k and $\bar{k}$ are integers, then

(i) $a \mid a$	(ii) $1 \mid a$	(iii) $a \mid 0$
(iv) $a \mid b \wedge b \mid c \Rightarrow a \mid c$	(v) $a \mid b \Rightarrow a \mid b\,c$	(vi) $a \mid b \wedge b \mid a \Rightarrow b = \pm a$

Proof

(i) $a = a.\,1 \Rightarrow a \mid a$

(ii) $a = 1\,.\,a \Rightarrow 1 \mid a$

(iii) $0 = a\,.\,0 \Rightarrow a \mid 0 \qquad (a \neq 0)$

(iv) $a \mid b \Rightarrow b = a\,k \qquad (1)$

$b \mid c \Rightarrow c = b\,\bar{k} \quad (2)$

from (1) and (2) we find

$c = a\,k\,\bar{k} \Rightarrow c = a\,(k\,\bar{k}) \Rightarrow a \mid c$

(v) $a \mid b \Rightarrow b = a\,k \qquad (3)$

multiply both sides of (3) by c

$b\,c \Rightarrow a\,k\,c \Rightarrow b\,c = (k\,c) \Rightarrow a \mid b\,c$

(vi) $a \mid b \Rightarrow b = a\, k$ (4)

$b \mid a \Rightarrow a = b\, \bar{k}$ (5)

from (4) and (5) we find $b = b\,(\bar{k}\, k) \Rightarrow k\, \bar{k} = 1$ $k = \pm 1$

$\therefore$ $b = \pm\, a$

Definition (2-2)

The integer p those divisors 1 and p is called prime number.

The set of prime numbers is {2, 3, 5, 7, …}.

Definition (2-3)

If $a \geq 1$ is not prime number, then a is called composite.

Theorem (2-4)

If p is a prime number and $p \mid m\, n$ then $p \mid m$ or $p \mid n$.

Proof

If $p \nmid m$ then $(p, m) = 1 \Rightarrow p\, x + my = 1$ (6)

$x, y \in z$

multiply both sides of (6) by n

$n\, p\, x + n\, m\, y = n$ (7)

since $p \mid m\, n \Rightarrow m\, n = p\, k$ (8)

from (7) and (8) we find

$n\, p\, x + p\, k\, y = n \Rightarrow n = p\,(n\, x + k\, y) \Rightarrow p \mid n$

Similarly:

$p \nmid n$ then $(p, n) = 1$ and with the same steps we find $p \mid m$

Theorem (2-5)

Every $n > 1$ has at least one prime divisor.

Proof

Suppose d is a least positive integer divides n where $d > 1$, if d is a prime the condition satisfied. If d is not prime, let $d = d_1\, d_2$ where $1<d_1<d$ and since $d>1$ is divides n, which contradiction that d is a least prime divisor of n.

The concept of congruence

Definition (2-6)

Let n is a positive integer. Then as integer a congruent to an integer b modulo n if n | (a-b) in symbols, we write $a \equiv b \pmod{n}$.

Theorem (2-7)

$a \equiv b \pmod{n}$ if and only if $a = b + n\,k$, $k \in z$.

Proof:

$a \equiv b \pmod{n} \Rightarrow n \mid (a - b) \Rightarrow a - b = n\,k \Rightarrow a = b + n\,k$

conversely

if $a = b + n\,k \Rightarrow a - b = n\,k \Rightarrow n \mid (a - b)$ so that $a \equiv b \pmod{n}$.

Theorem (2-8)

If n is a positive integer a, b, c $\in \mathbf{Z}$

(i) $a \equiv a \pmod{n}$ (Reflexive property)

(ii) $a \equiv b \pmod{n}$, $b \equiv a \pmod{n}$ (Symmetric property)

(iii) $a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$, $b \equiv c \pmod{n}$ (Transitive property)

(The congruence rela1tion is an equivalence)

Proof

(i) $a \equiv a \pmod{n} \Rightarrow n \mid (a - a) \Rightarrow n \mid 0$

since $n \mid 0 \Rightarrow n \mid (a - a) \Rightarrow a \equiv a \pmod{n}$

(ii) $a \equiv b \pmod{n} \Rightarrow n \mid (a - b) \Rightarrow a - b = nk,\ k \in \mathbf{Z}$

$-(b - a)\ nk \Rightarrow b - a = n(-k)\ n \mid (a - b) \Rightarrow b \equiv a \pmod{n}$

(iii) $a \equiv b \pmod{n} \Rightarrow a - b \Rightarrow nk,\ k \in \mathbf{Z}$ (9)

$b \equiv c \pmod{n} \Rightarrow b - c \Rightarrow n\bar{k},\ k \in \mathbf{Z}$ (10)

add (9) + (10)

$a - c = n(k + \bar{k}) \Rightarrow n \mid (a - c) \Rightarrow a = c \pmod{n}$

congruence relation is (reflexive, symmetric, transitive). So that it's equivalence relation).

Theorem (2-9)

$a \equiv b \pmod{n}$ if and only if a and b have the same reminder when divided by n.

Proof

$a \equiv b \pmod{n} \Rightarrow a - b \Rightarrow a = b + nk,\ \ k \in \mathbf{Z}$

Form division algorithm $b = n\,q + r \quad 0 \leq r < b$

$$a = b + nk = n\,q + r + nk = n\,(q + k) + r$$

therefore, a, b has the same reminder r when divided by n.

-conversely suppose both a and b have same remainder r when divided

From division algorithm

$$a = n\,q + r \qquad (11)$$

$$b = n\,\overline{q} + r \qquad (12)$$

subtract (11) – (12)

$$a - b = n\,q - n\,\overline{q} = n\,(q - \overline{q}) \Rightarrow n \mid (a - b)$$

$\therefore \quad a \equiv b \pmod{n}$.

Divisibility Test for (7, 13, 17)

Divisibility Test for 7

- Double the last digit of the number.
- Subtract it form the rest of the number.
- Repeat these two steps to find one- or two-digits' number and check it, to see if divisible by 7.

Example (3-1)

Check whether the number 24521 is divisible by 7.

Solution:

$$\begin{array}{r} 24521 \\ \underline{\quad 2} \\ 2450 \\ \underline{\quad 0} \\ 245 \\ \underline{10} \\ 14 \end{array}$$

Since $7 \mid 14 \Rightarrow 7 \mid 24521$

Other solution

Multiply digits of number beginning form the last digit by 1, 3, 2, 6, 4 and 5 respectively and find the sum to check it, if is divisible by 7.

$1 \times 1 + 3 \times 2 + 2 \times 5 + 6 \times 4 + 4 \times 2 = 1 + 6 + 10 + 24 + 8 = 49$

Since $7 \mid 49 \Rightarrow 7 \mid 24521$

Divisibility Test for 13

- Multiple the last digit of the number by 4.
- Add it to the rest of the number.
- Repeat these two steps to find two digits' number and check it, to see whether the number is divisible by 13.

Example (3-2)

Prove that the number 957657 is divisible by 13.

Solution

$$\begin{array}{r} 957657 \\ \underline{28} \\ 95793 \\ \underline{12} \\ 9591 \\ \underline{4} \\ 963 \\ \underline{12} \\ 108 \\ \underline{32} \\ 42 \end{array}$$

$13 \mid 42 \Rightarrow 13 \mid 957657$

Example (3-3)

Check whether the number 395165 is divisible by 13.

Solution

$$
\begin{array}{r}
395265 \\
\underline{20} \\
39546 \\
\underline{24} \\
3978 \\
\underline{32} \\
429 \\
\underline{36} \\
78
\end{array}
$$

Since 13 | 78⇒13 | 395265

Divisibility Test for 17

- Multiple the last digit of the number by 5.
- Subtract it form the rest of the number.
- Repeat these two steps to find two digits' number and check it, to see whether the number is divisible by 17.

Example (3-4)

If the number 123352 is divisible by 17.

Solution

$$
\begin{array}{r}
123352 \\
\underline{10} \\
12325 \\
\underline{25} \\
1207 \\
\underline{35} \\
85
\end{array}
$$

Since 17 | 85⇒17 | 123352

Divisibility Test for (19, 23, 29)

Divisibility Test for 19

- Double the last digit of the number.
- Add it to the rest of the number.
- Repeat these two steps to find two digits' number and check it, to see if divisible by 7.

Example (4-1)

Prove that the number 955719 is divisible by 19 without doing division.

Solution:

$$\begin{array}{l} 955719 \\ \underline{\quad\;\, 18} \\ 95589 \\ \underline{\;\; 18} \\ 9576 \\ \underline{\;\; 12} \\ 969 \\ \underline{18} \\ 114 \\ \underline{8} \\ 19 \end{array}$$

$19 \mid 19 \Rightarrow 19 \mid 955719$

i. Divisibility Test for 23

- Multiply the last digit of the number by 7.
- Add it to the rest of the number.
- Repeat these two steps to find two digits' number and check it, to see whether the number is divisible by 23.

Example (4-2)

Check whether the number 74635 is divisible by 23.

Solution:

74635
35
———
7498
56
———
805
35
———
115
35
———
46

Since 23 | 46 ⇒ 23 | 744635

Divisibility Test for 29

- Multiply the last digit of the number by 3.
- Add it to the rest of the number.
- Repeat these two steps to find two digits' number and check it, if divisible by 29.

Example (4-3)

If the number 93235 is divisible by 29.

Solution:

93235
15
———
9338
24
———
957
21
———
116
18
———
29

Since 29 | 29⇒ 29 | 93235

Example (4-4)

Check whether the number 62557 is divisible by 29.

Solution:

$$\begin{array}{r} 62557 \\ 21 \\ \hline 6276 \\ 18 \\ \hline 645 \\ 15 \\ \hline 79 \end{array}$$

Since $29 \nmid 79 \Rightarrow 29 \nmid 62557$

Conclusion

We reached the importance of numbers theory which it plays significant role in developing of applied science and technology.

In summary for any prime p ≠ 2, 5 it is possible to determine a divisibility rule that is based on a trimming algorithm congruence relation.

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A sharp Trudinger-Moser Type Inequality for Unbounded Domains and Higher Order Derivatives

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Abstract

The Trudinger- Moser inequality states that for functions $u \in H_o^{1,n}(\Omega)$, of bounded domain Ω with $\int |\nabla u| \, dx \leq 1$ one has $\lim_{k \to +\infty} \int_{B_1} (e^{\beta u_k^{\frac{n}{n-1}}} - 1) \, dx \leq c \, |\Omega|$, with c independent of u. It is shown that for $n = 2$ the bound $c|\Omega|$ may be replaced by a uniform constant d independent of Ω if the Drichlet norm is replaced by the Sobolev norm.In this paper the results for $n > 2$ have been showed with a lower bound and gradient estimate.

Keywords: *Truding-Moser inequality, blow-up analysis, best constant, unbounded domain, Mathematics subject classification*.

متباينة ترودنقر. موزار الحادة للمجالات غير المحدودة وللمشتقات عالية المستوى

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المؤلف:

مُسْتَخْلَص

وُجد ان متباينة ترودنقر. موزار تقرر انه لكل الدوال $u \in H_o^{1,n}(\Omega),$ والتي مجالها Ω (وهو مجال محدود) مع التكامل $\int |\nabla u| dx \leq 1$ لها النهاية $\lim_{k \to +\infty} \int_{B_1} (e^{\beta u_k^{\frac{n}{n-1}}} - 1)\, dx \leq c\, |\Omega|$ عندما يكون c مستقل عن u. وهذا واضح عند n=2 أن الحد $c|\Omega|$ يمكن استبداله بالثابت المنتظم d مستقلا عن Ω. عند استبدال نظيم در شلت بنظيم سو بوليف.عُرضت في هذه الورقة نتائج من أجل n >2 مع الحد الأقل وتقديرات الانحدار.

كلمات مفتاحية:*متباينة ترودنقر. موزار، تحليل الانشطار، أفضل ثابت، المجال غير المحدود، تصنيف المواضيع الرياضية.*

Introduction

Let $H_o^{1,p}(\Omega), \Omega \subseteq R^n$, be the usual Sobolev space. *i.e.* the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{H^{1,p}(\Omega)} = \left(\int_\Omega \left(|\nabla u|^p + |u|^p\right) dx\right)^{\frac{1}{p}}.$$

It is well-known that

$$H_o^{1,p}(\Omega) \subset L^{\frac{pn}{n-p}}(\Omega) \quad if \quad 1 \le p < n$$

$$H_o^{1,p}(\Omega) \subset L^{\infty}(\Omega) \quad if \quad n < p$$

The case $p = n$ is the limit case of these embeddings and it is known that

$$H_o^{1,p}(\Omega) \subset L^{q}(\Omega) \quad for \quad n \le q < +\infty$$

When Ω is a bounded domain, we usually use the Drichlet norm $\|u\|_D =$ $\left(|\nabla u|^n dx\right)^{\frac{1}{n}}$ in place of $\|\cdot\|_{H^{1,n}}$. In this case we have the famous Trudinger-Moser inequality (see [11], [4], [9]) for the limit case $p = n$ which states that

$$\sup_{\|u\|_D \le 1} \int_\Omega \left(e^{\propto |u|^{\frac{n}{n-1}}} - 1\right) dx = c(\Omega, \propto) \begin{cases} < +\infty & when \quad \propto \le \propto_n \\ = +\infty & when \quad \propto > \propto_n \end{cases} \qquad (1)$$

where $\propto = n\, \omega_{n-1}^{\frac{1}{n-1}}$, and ω_{n-1} is the measure of unit sphere in R^n.The Trudinger- Moser result has been extended to sphere of higher order and Sobolev spaces over compact fields (see [7], [13]). Moreover, for any bounded Ω, the constant $c(\Omega, \propto)$ can be attained. For the attainability, we refer to[8], [5], [13] and (Li, 2001).

Another interesting extension of (1) is to construct Trudinger-Moser type inequalities on unbounded domains. When $n = 2$, this has been done in (Ruf, 2005). On the other hand, for unbounded domain in R^n.

Let

$$\Phi(t) = e^t - \sum_{j=1}^{n-2} \frac{t^i}{j|}$$

The result in (Li and Ruf, 2000) says that

Theorem C. For any $\propto \in (0, \propto_n)$ there is a constant $C(\propto)$ such that

$$\int_{R^n} \Phi\left(\propto \left(\frac{|u|}{\|\nabla u\|_{L^n(R^n)}} \right)^{\frac{n}{n-1}} \right) dx \le C(\propto) \frac{\|u\|^n_{L^n(R^n)}}{\|\nabla u\|^n_{L^n(R^n)}}, \quad for \quad u \in H^{1,n}(R^n) \setminus \{0\}. \qquad (2)$$

We shall discuss the critical case $\propto = \propto_n$. More precisely, we prove the following:

Theorem (1.1) (Adachi and Tanaka, 1999). There exists a constant $d > 1$, such that, for any domain $\Omega \subset R^n$,

$$\sup_{u \in H^{1,n}(\Omega), \|u\|_{H^{1,n}(\Omega)}} \int_\Omega \Phi\left(\propto_n |u|^{\frac{n}{n-1}} \right) dx \le d. \qquad (3)$$

The inequality is sharp: for any $\propto > \propto_n$, the supremum is $+\infty$.

We set

$$S = \sup_{u \in H^{1,n}(R^n), \|u\|_{H^{1,n}(R^n)}} \int_{R^n} \Phi\left(\propto_n |u|^{\frac{n}{n-1}} \right) dx.$$

Further, we will prove:

Theorem (1.2) (Ruf, 2005). S is attained. In other words, we can find a function $u \in H^{1,n}(R^n)$, WITH $\|u\|_{H^{1,n}(R^n)}$=1 such that

$$S = \int_{R^n} \Phi\left(\propto_n |u|^{\frac{n}{n-1}} \right) dx.$$

The second part of Theorem (1.2) is trivial. Given any fixed $\propto > \propto_n$, we take $\beta \in (\propto_n, \propto)$. By (1) we can find a positive sequence $\{ u_k \}$ in

$$\left\{ u \in H_0^{1.n}(B_1) : \int_{B_1} |\nabla u|^n dx = 1 \right\},$$

such that

$$\lim_{k \to +\infty} \int_{B_1} e^{\beta u_k^{\frac{n}{n-1}}} = +\infty.$$

By Lion's Lemma, we get $u_k \to 0$. Then by compact embedding theorem, we may assume $\|u_k\|_{L^p(B_1)} \to 0$ for any $p > 1$. Then, $\int_{R^n} \left(|\nabla u_k|^n + |u_k|^n\right) dx \to 1$, and

$$\propto \left(\frac{u_k}{\|u_k\|_{H^{1,n}}}\right)^{\frac{n}{n-1}} > \beta u_k^{\frac{n}{n-1}}.$$

When k is sufficiently large. So, we get

$$\lim_{k\to+\infty} \int_{R^n} \Phi(\propto \left(\frac{u_k}{\|u_k\|_{H^{1,n}}}\right)^{\frac{n}{n-1}})\, dx \geq \lim_{k\to+\infty} \int_{B_1} (e^{\beta u_k^{\frac{n}{n-1}}} - 1)\, dx = +\infty.$$

The first part of Theorem (1.1) and Theorem (1.2) will be proved by blow up analysis. We will use the ideas from [14] and (Li, 2005). However, in the unbounded case we do not obtain the strong convergence of u_k in $L^n(R^n)$, and so we have more techniques.

Concretely we will find positive and symmetric functions $u_k \in H^{1,n}(B_{R_k})$ which satisfy

$$\int_{B_{R_k}} \left(|\nabla u_k|^n + |u_k|^n\right) dx \to 1$$

and

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \sup_{\int_{B_{R_k}} \left(|\nabla v|^n + |v|^n\right) = 1, v \in H_0^{1,n}(B_{R_k})} \int_{B_{R_k}} \Phi\left(\beta_k |v|^{\frac{n}{n-1}}\right) dx.$$

Here, β_k is an increasing sequence tending to $\propto_n$, and R_k is an increasing sequence tending to $+\infty$.

Further, u_k satisfies the following equation

$$-div|\nabla u_k|^{n-2} \nabla u_k + u_k^{n-1} = \frac{u_k^{\frac{1}{n-1}} \Phi'\left(\beta_k u_k^{\frac{n}{n-2}}\right)}{\lambda_k},$$

where λ_k is Lagrange multiplier.

Then, there are two possibilities. If $c_k = \max u_k$ is bounded from above, then it is easy to see that

$$\lim_{k\to+\infty} \int_{R^n} \left(\Phi\left(\beta_k u_k^{\frac{n}{n-2}}\right) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!}\right) dx = \int_{R^n} \left(\Phi\left(\propto_k u^{\frac{n}{n-2}}\right) - \frac{\propto_k^{n-1} u^n}{(n-1)!}\right) dx,$$

where u is the weak limit of $c_k = \max u_k$. It then follows that either $\int_{\mathcal{R}^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx$ converges to

$\int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx$ or $S \le \frac{\propto_n^{n-1}}{(n-1)!}$.

If c_k is not bounded, the key point of the proof is to show that

$$\frac{n}{n-1}\beta_k c_k^{\frac{1}{n-1}}\left(u_k(r_k x) - c_k\right) \to -n\log\left(1 + c_n^{\frac{n}{n-1}}\right),$$

locally for a suitably chosen sequence r_k (and with $c_n = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n-1}}$), and that

$$c_k^{\frac{1}{n-1}} u_k \to G,$$

On any $\Omega \subset\subset R^n \setminus \{0\}$, where G is some Green function.

In section (5.2), we will construct a function sequence $u_\in$ such that

$$\int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx > \frac{\omega_{n-1}}{n} e^{\propto_n A + 1 + 1/2 + \cdots + 1/(n-1)}$$

when $\in$ is sufficiently small. And also, we construct, for $n > 2$, a function sequence $u_\in$ such that for $\in$ sufficiently small

$$\int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx > \frac{\propto_n^{n-1}}{(n-1)!}.$$

Thus, together with Ruff's result of attainability in [14] for the case $n = 2$, we will get Theorem (1.2).

Definition (1.3) (Li and Ruf, 2000). To define the maximizing sequence, let $\{R_k\}$ be an increasing sequence which diverges to infinity, and $\{B_k\}$ an increasing sequence which converges to α_n. By compactness, we can find positive functions $u \in H^{1,n}(B_{R_k})$ with $\int_{B_{R_k}} \left(|\nabla u_k|^n + |u_k|^n\right) dx = 1$ such that

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \sup_{\int_{B_{R_k}}\left(|\nabla v|^n + |v|^n\right) = 1, v \in H_0^{1,n}(B_{R_k})} \int_{B_{R_k}} \Phi\left(\beta_k |v|^{\frac{n}{n-1}}\right) dx.$$

Moreover, we may assume $\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx = \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx$ is increasing.

Lemma (1.4) (Li and Ruf, 2000). Let u_k as above. Then

(a) u_k is a maximizing sequence for S;

(b) u_k may be chosen to be radially symmetric and decreasing.

Proof. (a) Let η be a cut-off function which is 1 on B_1 and 0 on $R^n \setminus B_2$. Then given any $\varphi \in H^{1,n}(R^n)$ with $\int_{R^n} \left(|\nabla \varphi|^n + |\varphi|^n \right) dx = 1$, we have

$$\tau^n(L) := \int_{R^n} \left(\left| \nabla \eta\left(\frac{x}{L}\right) \varphi \right|^n + \left| \eta\left(\frac{x}{L}\right) \varphi \right|^2 \right) dx \to 1, \quad as \quad L \to +\infty.$$

Hence, for a fixed L and $R_k > 2L$

$$\int_{B_L} \Phi\left(\beta_k \left| \frac{\varphi}{\tau(L)} \right|^{\frac{n}{n-1}} \right) dx \leq \int_{B_{2L}} \Phi\left(\beta_k \left| \frac{\eta\left(\frac{x}{L}\right)\varphi}{\tau(L)} \right|^{\frac{n}{n-1}} \right) dx \leq \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx .$$

By the Levi Lemma, we then have

$$\int_{B_L} \Phi\left(\propto_n \left| \frac{\varphi}{\tau(L)} \right|^{\frac{n}{n-1}} \right) dx \leq \lim_{k \to +\infty} \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx .$$

Then, letting $L \to +\infty$, we get

$$\int_{R^n} \Phi\left(\propto_n |\varphi|^{\frac{n}{n-1}} \right) dx \leq \lim_{k \to +\infty} \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx .$$

Hence, we get

$$\lim_{k \to +\infty} \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx = \sup_{\int_{R^n} \left(|\nabla v|^n + |v|^n \right) = 1, v \in H_0^{1,n}\left(B_{R_k}\right)} \int_{R^n} \Phi\left(\propto_n |v|^{\frac{n}{n-1}} \right) dx.$$

(b) Let u_k^* be the radial rearrangement of u_k, then we have

$$\tau_k^n := \int_{B_{R_k}} \left(\left| \nabla u_k^* \right|^n + u_k^{*n} \right) dx \leq \int_{B_{R_k}} \left(\left| \nabla u_k \right|^n + u_k^n \right) dx = 1.$$

It is well-known that $\tau_k = 1$ if and only if u_k is radial. Since

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{*\frac{n}{n-1}}\right)dx = \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx ,$$

we have

$$\int_{B_{R_k}} \Phi\left(\beta_k \left(\frac{u_k^*}{\tau_k}\right)^{\frac{n}{n-1}}\right)dx \geq \int_{B_{R_k}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx ,$$

And $"="$ holds if and only if $\tau_k = 1$. Hence $\tau_k = 1$ and

$$\int_{B_{R_k}} \Phi\left(\beta_k u_k^{*\frac{n}{n-1}}\right)dx = \sup_{\int_{R^n}\left(|\nabla v|^n + |v|^n\right) = 1, v\in H_0^{1,n}\left(B_{R_k}\right)} \int_{B_{R_k}} \Phi\left(\propto_n |v|^{\frac{n}{n-1}}\right)dx.$$

So, we can assume $u_k = u_k(|x|)$, and $u_k(r)$ is decreasing

Assume now $u_k = u$. Then, to prove Theorem (1.1) and Theorem (1.2), we only need to show that

$$\lim_{k\to+\infty} \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx = \int_{R^n} \Phi\left(\propto_n u_k^{\frac{n}{n-1}}\right)dx .$$

Definition (1.5) (Li and Ruf, 2000). By the definition of u_k, we have the equation

$$-div|\nabla u_k|^{n-2}\nabla u_k + u_k^{n-1} = \frac{u_k^{\frac{1}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)}{\lambda_k} , \qquad (4)$$

where λ_k is the constant satisfying

$$\lambda_k = \int_{B_{R_k}} u_k^{\frac{n}{n-2}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx .$$

First, we need the following:

Lemma (1.6) (Li and Ruf, 2000). $\inf_k \lambda_k > 0$.

Proof. Assume $\lambda_k \to 0$. Then

$$\int_{R^n} u_k^n\, dx \leq C\int_{R^n} u_k^{\frac{n}{n-2}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \leq C\lambda_k \to 0 .$$

Since $u_k(|x|)$ is decreasing, we have $u_k^n(L)|B_L| \leq \int_{B_L} u_k^n \leq 1$, and then

$$u_k(L) \leq \frac{n}{\omega_n L^n} . \qquad (5)$$

set $\in=\dfrac{n}{\omega_n L^n}$. Then $u_k(x)\le\in$ for any $x\notin B_L$, and hence, we have, using the form of Φ, that

$$\lambda_k=\int_{R^n\setminus B_L}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx\le C\int_{R^n\setminus B_L}u_k^n\,dx\le C\lambda_k\to 0.$$

And on B_L, since $u_k\to 0$ in $L^q(B_L)$ for any $q>1$, we have by Lebesgue

$$\lim_{k\to+\infty}\int_{B_L}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \le \lim_{k\to+\infty}\left[\int_{B_L}Cu_k^{\frac{n}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx+\int_{\{x\in B_L:u_k(x)\le 1\}}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx\right]$$

$$\le \lim_{k\to+\infty}C\lambda_k+\int_{B_L}\Phi(0)dx=0$$

This is impossible. □

Results(1.7) (Shawgy and Mahgoub, 2011): (i) Definition (1.5) and results (2.8) implies that if we set $v_g=x$ and $\varphi(z)=z_e$ then we have

$$\Phi\left(B_k\,|u_k|^{\frac{n}{n-1}}\right)=e^{p_k|u_k|^{n/n-1}}.$$

(ii) Theorem (1.2) shows that

$$\propto\left(\frac{u_k}{\|u_k\|_{H^{1,n}}}\right)^{\frac{n}{n-1}}>\beta\,u_k^{\frac{n}{n-1}}.$$

$$\|u_k\|^{n/n-1}<\frac{\alpha^{\frac{n}{n-1}}}{\beta}\qquad\text{or}\qquad\|u_k\|<\frac{\alpha}{\beta^{n/n-1}}.$$

If $u_k\to u\in H^{1,n}(R^n)$ $with$ $\|u\|_{H^{1,n}(R^n)}=1$ then $\beta<\alpha^{n/n-1}$.

(iii) If $c_k=Max\,u_k$, where u is the weak limit of $c_k=Max\,u_k$, it follows that

$$S\le\frac{\beta}{(n-1)1}\|u_k\|,\ n>1.$$

We denote $c_k=\max u_k=u_k(0)$. Then we have

Lemma (1.8) (Li and Ruf, 2000). If $\sup\limits_k c_k<+\infty$,

(i) Theorem (5.1.1) holds;

(ii) if S is not attained, then

$$S \leq \frac{\propto_n^{n-1}}{(n-1)!}.$$

Proof. If $\sup_k c_k < +\infty$, then $u_k \to u$ in $C^1_{10c}(R^n)$. By (5), we are able to find L $s.t$ $u_k(x) \leq \in$ for $x \notin B_L$. Then

$$\int_{R^n \setminus B_L} (\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) - \frac{B_k^{n-1} u_k^n}{(n-1)!})dx \leq C\int_{R^n \setminus B_L} u_k^n \, dx \leq C \in^{\frac{n^2}{n-1}-n} \int_{R^n} u_k^n \, dx \leq C \in^{\frac{n^2}{n-1}-n}.$$

Letting $\in \to 0$, we get

$$\int_{R^n \setminus B_L} (\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) - \frac{B_k^{n-1} u_k^n}{(n-1)|})dx = \int_{R^n \setminus B_L} (\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n^{n-1} u^n}{(n-1)!})dx.$$

Hence

$$\lim_{k\to+\infty} \int_{R^n \setminus B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \int_{R^n} \Phi\left(\propto_n u_k^{\frac{n}{n-1}}\right) dx + \frac{\propto_n^{n-1}}{(n-1)!} \lim_{k\to+\infty} \int_{R^n} \left(u_k^n - u^n\right) dx. \qquad (6)$$

When $u = 0$, we can denote from (6) that

$$S \leq \frac{\propto_n^{n-1}}{(n-1)!}.$$

Now, we assume $u \neq 0$. Set

$$\tau^n = \lim_{k\to+\infty} \frac{\int_{R^n} u_k^n dx}{\int_{R^n} u^n dx}.$$

By the Levi Lemma, we have $\tau \geq 1$.

Let $\bar{u} = u\left(\frac{x}{\tau}\right)$. Then, we have

$$\int_{R^n} |\nabla \bar{u}|^n dx = \int_{R^n} |\nabla u|^n dx \leq \lim_{k\to+\infty} \int_{R^n} |\nabla u_k|^n dx,$$

and

$$\int_{R^n} \bar{u}^n dx = \tau^n \int_{R^n} u^n dx = \lim_{k\to+\infty} \int_{R^n} u_k^n dx.$$

Then

$$\int_{R^n} \left(|\nabla \bar{u}|^n + \bar{u}^n\right) dx \leq \lim_{k\to+\infty} \int_{R^n} \left(|\nabla u_k|^n + \left|u_k^n\right|^n\right) dx = 1.$$

Hence, we have by (6)

$$S \geq \int_{R^n} \Phi\left(\propto_n \bar{u}^{\frac{n}{n-1}} \right) dx$$

$$= \tau^n \int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}} \right) dx$$

$$= [\int_{R^n} \Phi\left(\propto_n \bar{u}^{\frac{n}{n-1}} \right) dx + \left(\tau^n - 1\right) \int_{R^n} \frac{\propto_n^{n-1}}{(n-1)!} dx] + \left(\tau^n - 1\right) \int_{R^n} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n}{(n-1)!} \right) dx$$

$$= \lim_{k \to +\infty} \int_{R^n \setminus B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}} \right) dx + \left(\tau^n - 1\right) \int_{R^n} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n}{(n-1)|} \right) dx$$

$$= S + \left(\tau^n - 1\right) \int_{R^n} \left(\Phi\left(\propto_n u^{\frac{n}{n-1}}\right) - \frac{\propto_n}{(n-1)!} \right) dx .$$

Since $\Phi\left(\propto_n' u^{\frac{n}{n-1}}\right) - \frac{\propto_n^{n-1}}{(n-1)!} u^n > 0$, we have $\tau = 1$, and then

$$S = \int_{R^n} \Phi\left(\propto_n u^{\frac{n}{n-1}}\right) dx .$$

So, u is an extremal function.

From now on, we assume $c_k \to +\infty$. We perform blow up procedure:

$$\tau_k^n = \frac{\lambda_k}{c_k^{\frac{n}{n-1}} e^{\beta_k c_k^{\frac{n}{n-1}}}} .$$

By (5) we can find a sufficiently L such that $u_k \leq 1$ on $R^n \setminus B_L$. Then

$$\int_{B_L} \left| \nabla\left(u_k - u_k(L)^+\right) \right|^n dx \leq 1 ,$$

and hence by (1), we have

$$\int_{B_L} e^{\propto_n \left| \left(u_k - u_k(L)^+\right) \right|^{\frac{n}{n-1}}} \leq C(L) .$$

Clearly, for any $p < \propto_n$ we can find a constant $C(p)$, such that

$$p u_k^{\frac{n}{n-1}} \leq \propto_n \left| u_k - u_k(L)^+ \right|^{\frac{n}{n-1}} + C(p) ,$$

and then, we get

$$\int_{B_L} e^{p u_k^{\frac{n}{n-1}}} dx \le C = C(L, p).$$

Hence

$$\lambda_k e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}} = e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}} \left[\int_{R^n \setminus B_L} u_k^{\frac{n}{n-1}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + \int_{B_L} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \right]$$

$$\le C \int_{R^n \setminus B_L} u_k^n \, dx \; e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}} + \int_{B_L} e^{\frac{\beta_k}{2} u_k^{\frac{n}{n-1}}} u_k^{\frac{n}{n-1}} \, dx.$$

Since u_k converges in $L^q(B_L)$ for any $q > 1$, we get $\lambda_k \le C e^{\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}}$ and hence

$$r_k^n \le C e^{-\frac{\beta_k}{2} c_k^{\frac{n}{n-1}}}.$$

Now, we set

$$v_k(x) = u_k(r_k x), \quad w_k(x) = \frac{n}{n-1} \beta_k C_k^{\frac{1}{n-1}} (v_k - c_k),$$

where v_k and w_k are defined on $\Omega_k = \{x \in R^n : r_k x \in B_1\}$. Using the definition of r_k^n and (4) we have

$$-div\left|\nabla w_k\right|^{n-2} \nabla w_k = \frac{v_k^{\frac{1}{n-1}}}{c_k^{\frac{1}{n-1}}} \left(\tfrac{n}{n-1} \beta_k\right)^{n-1} e^{\beta_k \left(v_k^{\frac{n}{n-1}} - c_k^{\frac{n}{n-1}}\right)} + O\left(r_k^n c_k^n\right).$$

In [9], we know that $osc_{B_R} w_k \le C(R)$ for any $R > 0$. Then from the result in (Dibendetto, 1983) (or [8]), it follows that $\|w_k\|_{C^{1,\delta}(B_R)} < C(R)$. Therefore w_k converges in C^1_{l0c} and $v_k - c_k \to 0$ in C^1_{l0c}.

Since

$$v_k^{\frac{n}{n-1}} = c_k^{\frac{n}{n-1}} \left(1 + \frac{v_k - c_k}{c_k}\right)^{\frac{n}{n-1}} = c_k^{\frac{n}{n-1}} \left(1 + \frac{n}{n-1} \frac{v_k - c_k}{c_k} + O\left(\frac{1}{c_k^2}\right)\right),$$

we get $\beta_k \left(v_k^{\frac{n}{n-1}} - c_k^{\frac{n}{n-1}}\right) \to w$ in C^1_{l0c}, and so we have

$$-div\left|\nabla w\right|^{n-2} \nabla w = \left(\frac{n \propto_n}{n-1}\right)^{n-1} e^w, \qquad (7)$$

with

$$w(0) = 0 = \max w.$$

Since w is radially symmetric and decreasing, it is easy to see that (7) has only one solution. We can check that

$$w(x) = -n\log\left(1 + c_n |x|^{\frac{n}{n-1}}\right), \quad \text{and} \quad \int_{R^n} e^w dx = 1,$$

where $c_n = \left(\frac{w_{n-1}}{n}\right)^{\frac{1}{n-1}}$. Then,

$$\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{B_{Lr_k}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx = \lim_{L\to+\infty} \int_{R^n} e^w dx = 1. \qquad (8)$$

For $A > 1$, let $u_k^A = \min\{u_k, \frac{c_k}{A}\}$. We have

Lemma (1.9) (Li and Ruf, 2000). For any $A > 1$, there holds

$$\limsup_{k\to+\infty} \int_{R^n} \left(\left|\nabla u_k^A\right|^n + \left|u_k^A\right|^n \right) dx \le \frac{1}{A}. \qquad (9)$$

Proof. Since $\left|\left\{x : u_k \ge \frac{c_k}{A}\right\}\right| \left|\frac{c_k}{A}\right|^n \le \int_{\left\{u_k \ge \frac{c_k}{A}\right\}} u_k^n \le 1$, we can find a sequence $\rho_k \to 0$ such that

$$\{x : u_k \ge \frac{c_k}{A}\} \subset B_{\rho_k}.$$

Since u_k converges in $L^p(B_1)$ for any $p > 1$, we have

$$\lim_{k\to+\infty} \int_{\left\{u_k \ge \frac{c_k}{A}\right\}} \left|u_k^A\right|^p dx \le \lim_{k\to+\infty} \int_{\left\{u_k \ge \frac{c_k}{A}\right\}} u_k^p \, dx = 0.$$

and

$$\lim_{k\to+\infty} \int_{R^n} \left(u_k - \frac{c_k}{A}\right)^+ u_k^p \, dx = 0$$

for any $p > 0$.

Hence, testing equation (4) with $\left(u_k - \frac{c_k}{A}\right)^+$, we have

$$\int_R \left(\left|\nabla \left(u_k - \frac{c_k}{A}\right)^+\right|^n + \left(u_k - \frac{c_k}{A}\right)^+ u_k^{n-1} \right) dx = \int_{R^n} \left(u_k - \frac{c_k}{A}\right)^+ \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx + o(1)$$

$$\ge \int_{B_{Lr_k}} \left(u_k - \frac{c_k}{A}\right)^+ \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx + o(1)$$

$$= \int_{B_L} \frac{v_k - \frac{c_k}{A}}{c_k} \left(\frac{v_k - \frac{c_k}{A}}{c_k} + 1\right)^{\frac{1}{n-1}} e^{w_k + o(1)} dx + o(1).$$

Hence

$$\liminf_{k\to+\infty}\int_R\left(\left|\nabla\left(u_k-\tfrac{c_k}{A}\right)^+\right|^n+\left(u_k-\tfrac{c_k}{A}\right)^+u_k^{n-1}\right)dx\geq\frac{A-1}{A}\int_{B_L}e^{w}dx.$$

Letting $L\to+\infty$, we get

$$\liminf_{k\to+\infty}\int_R\left(\left|\nabla\left(u_k-\tfrac{c_k}{A}\right)^+\right|^n+\left(u_k-\tfrac{c_k}{A}\right)^+u_k^{n-1}\right)dx\geq\frac{A-1}{A}.$$

Now, observe that

$$\int_{R^n}\left(\left|\nabla u_k^A\right|^n+\left|u_k^A\right|^n\right)dx= \quad 1-\int_R\left(\left|\nabla\left(u_k-\tfrac{c_k}{A}\right)^+\right|^n+\left(u_k-\tfrac{c_k}{A}\right)^+u_k^{n-1}\right)dx \quad +\int_{R^n}\left(u_k-\tfrac{c_k}{A}\right)^+dx- \quad \int_{\left\{u_k\geq\frac{c_k}{A}\right\}}u_k^n$$

$$\int_{\left\{u_k\geq\frac{c_k}{A}\right\}}\left|u_k^A\right|^n dx$$

$$\leq 1-\left(1-\tfrac{1}{A}\right)+o(1).$$

Hence, we get this Lemma

Corollary (1.10) (Li and Ruf, 2000). We have

$$\int_{R^n\setminus B_\delta}\left(\left|\nabla u_k^A\right|+u_k^n\right)dx=0,$$

for any $\delta>0$, and then $u=0$.

Proof. Letting $A\to+\infty$, then for any constant c, we have

$$\int_{\{u_k\leq c\}}\left(\left|\nabla u_k^A\right|+u_k^n\right)dx\to 0.$$

So, we get this Corollary.

Lemma (1.11) (Li and Ruf, 2000). We have

$$\lim_{k\to+\infty}\int_{R^n\setminus B_L}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx \leq \lim_{L\to+\infty}\lim_{k\to+\infty}\left(e^{\beta_k u_k^{\frac{n}{n-1}}}-1\right)dx=\limsup_{k\to+\infty}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}. \qquad (10)$$

and consequently

$$\frac{\lambda_k}{c_k},\quad \text{and}\quad \sup_k\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}<+\infty. \qquad (11)$$

Proof. We have

$$\int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \int_{\left\{u_k \le \frac{c_k}{A}\right\}} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + \int_{\left\{u_k \le \frac{c_k}{A}\right\}} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \int_{R^n} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}}$$

$$\int_{R^n} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \ .$$

Applying (5), we can find L such that $u_k \le 1$ on $R^n \setminus B_L$. Then by Corollary (1.11) and the form of Φ, we have

$$\lim_{k\to+\infty} \int_{R^n \setminus B_l} \Phi\left(p\beta_k (u_k^A)^{\frac{n}{n-1}}\right) dx \le \lim_{k\to+\infty} C(p) \int_{R^n \setminus B_l} u_k^n \, dx = 0 \qquad (12)$$

for any $p > 0$.

Since by Lemma (1.10) $\limsup_{k\to+\infty} \int_{R^n} \left(\left|\nabla u_k^A\right|^n + \left|u_k^A\right|^n \right) dx \le \frac{1}{A} < 1$, it follows from (1) that

$$\sup_k \int_{B_L} e^{p'\beta_k \left(u_k^A - u_k(L)^+\right)^{\frac{n}{n-1}}} dx < +\infty$$

for any $p' < A^{\frac{1}{n-1}}$. Since for any $p < p'$

$$p\left(u_k^A\right)^{\frac{n}{n-1}} \le p'\left(\left(u_k^A - u_k(L)^+\right)^{\frac{n}{n-1}} + C(p, p') \right),$$

we have

$$\sup_k \int_{B_L} \Phi\left(\beta_k \left(u_k^A\right)^{\frac{n}{n-1}}\right) dx < +\infty \qquad (13)$$

for any $p < A^{\frac{1}{n-1}}$. Then on B_L, by the weak compactness of Banach space, we get

$$\lim_{k\to+\infty} \int_{B_L} \Phi\left(\beta_k \left(u_k^A\right)^{\frac{n}{n-1}} \right) dx = \int_{B_L} \Phi(0) dx = 0 .$$

Hence, we have

$$\lim_{k\to+\infty} \int_{B_L} \Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \lim_{L\to+\infty} \lim_{k\to+\infty} A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}} \int_{B_L} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx + C_\in = \lim_{K\to+\infty} A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}} + C_\in .$$

As $A \to 1$ and $\in \to 0$, we obtain (10).

If $\frac{\lambda_k}{c_k}$ was bounded or $\sup_k \frac{c_k^{\frac{n}{n-1}}}{\lambda_k} = +\infty$, it would follow from (10) that

$$\sup_{\int_{R^n}\left(|\nabla v|^n + |v|^n\right)=1, v\in H_0^{1,n}\left(B_{R_k}\right)} \int_{B_{R_k}} \Phi\left(\propto_n |v|^{\frac{n}{n-1}}\right) dx = 0 .$$

Which is impossible.

Lemma (1.12) (Carleson and Chang, 1986). We have that $c_k \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)$ converges to δ_0 weakly, *i.e.* for any $\varphi \in D\left(R^n\right)$

$$\lim_{k\to+\infty} \varphi c_k \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} \int_{B_L} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \varphi(0).$$

Proof. Suppose $supp\, \varphi \subset B_p$. We split the integral

$$\int_{B_p} \varphi \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \int_{\left\{u_k \ge \frac{c_k}{\lambda}\right\} \setminus B_{Lr_k}} \cdots + \int_{B_{Lr_k}} \cdots + \int_{\left\{u_k < \frac{c_k}{\lambda}\right\}} \cdots = I_1 + I_2 + I_3 .$$

We have

$$I_1 \le A\|\varphi\|_{C^0} \int_{R^n \setminus B_{Lr_k}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = A\|\varphi\|_{C^0} \left(1 - \int_{B_L} e^{w_k + o(1)} dx\right),$$

and

$$I_2 = \int_{B_L} \varphi(r_k x) \frac{c_k \left(c_k + \left(v_k - c_k\right)\right)^{\frac{1}{n-1}}}{c_k^{\frac{n}{n-1}}} e^{w_k + o(1)} dx = \varphi(0) \int_{B_L} e^{w} dx + o(1) = \varphi(0) + o(1).$$

By (12) and (13) we have

$$\int_{R^n} \Phi\left(p\beta_k\, | u_k^A |^{\frac{n}{n-1}}\right) dx < C$$

for any $p < A^{\frac{1}{n-1}}$. We set $\frac{1}{q} + \frac{1}{p} = 1$. Then we get by (11)

$$I_3 = \int_{\left\{u_k \le \frac{c_k}{\lambda}\right\}} \varphi c_k \frac{u_k^{\frac{1}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx \le \frac{c_k}{\lambda_k} \|\varphi\|_{C^0} \left\| u_k^{\frac{1}{n-1}} \right\|_{L^q\left(R^n\right)} \left\| e^{\beta_k \left|u_k^A\right|^{\frac{n}{n-1}}} \right\|_{L^q\left(R^n\right)} \to 0 .$$

Letting $L \to +\infty$, we deduce now that

$$\lim_{k\to+\infty} \int_{R^n} \varphi \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx = \varphi(0)$$

Proposition (1.13) (Li and Ruf, 2000). On any $\Omega \subset\subset R^n \setminus \{0\}$, we have that $c_k^{\frac{1}{n-1}} u_k$ converges to G in $C'(\Omega)$, where $G \in C_{loc}^{1,\propto}(R^n \setminus \{0\})$ satisfies the following equation

$$-div|\nabla G|^{n-2}\nabla G + G^{n-1} = \delta_0 . \tag{14}$$

Proof. We set $U_k = c_k^{\frac{1}{n-1}} u_k$, which satisfy by (4) the equations:

$$-div|\nabla U_k|^{n-2}\nabla U + U_k^{n-1} = \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right). \tag{15}$$

For our purpose, we need to prove

$$\int_{B_R} |U_k|^q dx \le C(q,R),$$

where $C(q,R)$ does not depend on k. We use the idea in [80] to prove this statement.

Set $\Omega_t = \{0 \le U_k \le t\}, U_k^t = \min\{U_k, t\}$. Then we have

$$\int_{\Omega_t} \left(|\nabla U_k|^n + |U_k|^n\right) dx \le \int_{R^n} \left(-U_k^t \Delta_n U_k + U_k^t U_k^{n-1}\right) = \int_{R^n} U_k^t \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) \le 2t .$$

Let η be a radially symmetric cut-off function which is 1 on B_R and 0 on B_{2R}^c. Then,

$$\int_{B_{2R}} \left|\nabla \eta U_k^t\right|^n dx \le C_1(R) + C_2(R)t .$$

Then, when t is bigger than $\frac{C_1(R)}{C_2(R)}$, we have

$$\int_{B_{2R}} \left|\nabla \eta U_k^t\right|^n dx \le 2C_2(R)t .$$

Set ρ such that $U_k(\rho) = t$. Then we have

$$\inf\{\int_{B_{2R}} |\nabla v|^n dx : v \in H_0^{1,n}(B_{2R}) \quad \text{and} \quad v|_{B_\rho} = t\} \le 2C_2(R)t.$$

On the other hand, the $\inf$ is achieved by $-t \log\frac{|x|}{2R} \Big/ \log\frac{2R}{\rho}$. By a direct computation, we have

$$\frac{\omega_{n-1} t^{n-1}}{\left(\log\frac{2R}{\rho}\right)^{n-1}} \le 2R ,$$

and hence for any $t > \frac{C_1(R)}{C_2(R)}$

$$\left|\{x \in B_{2R} : U_k \geq t\}\right| = \left|B_\rho\right| \leq C_3(R)e^{-A(R)t},$$

where $A(R)$ is a constant only depending on R. Then, for any $\delta < A$,

$$\int_{B_R} e^{\delta U_k} dx \leq \sum_{m=0}^{\infty} \mu(\{m \leq U_k \leq m+1\})e^{\delta(m+1)} \leq \sum_{m=0}^{\infty} e^{-(A-\delta)m} e^{\delta} \leq C.$$

Then, testing the equation (15) with the function $\log \frac{1+2(U_k - U_k(R))^+}{1-(U_k - U_k(R))^+}$, we get

$$\int_{B_R} \frac{|\nabla U_k|^n}{(1+U_k - U_k(R))(1+2U_k - 2U_k(R))} dx$$

$$\leq \log 2 \int_{B_R} \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right) dx - \int_{B_R} U_k \log \frac{1+2(U_k - U_k(R))^+}{1-(U_k - U_k(R))^+} dx \leq C.$$

Given $q < n$, by Young's inequality, we have

$$\int_{B_R} |\nabla U_k|^q dx \leq \int_{B_R} [\ \frac{|\nabla U_k|^n}{(1+U_k - U_k(R))(1+2U_k - 2U_k(R))} dx + ((1+U_k)(1+2U_k))^{\frac{n}{n-1}}\ ]$$

$$\leq \int_{B_R} [\ \frac{|\nabla U_k|^n}{(1+U_k - U_k(R))(1+2U_k - 2U_k(R))} dx + Ce^{\delta U_k}\ ] dx.$$

Hence, we are able to assume that U_k converges to a function G weakly in $H^{1,p}(B_R)$ for any R and $p < n$. Applying Lemma (1.12), we get (14).

Hence U_k is bounded in $L^q(\Omega)$ for any $q > 0$. By Corollary (1.10) and Theorem C, $e^{\beta_k u_k^{\frac{n}{n-1}}}$ is also bounded in $L^q(\Omega)$ for any $q > 0$. Then applying [9], and [8](or [5]), we get $\|U_k\|_{C^{1,\propto}(\Omega)} \leq C$. So U_k converges to G in $C^1(\Omega)$.

For the Green function G we have the following result.

Lemma (1.14) (Li and Ruf, 2000). $G \in C_{loc}^{1,\propto}(R^n \setminus \{0\})$ and near 0 we can write

$$G = -\frac{1}{\propto_n} \log r^n + A + O(r^n \log^n r);$$

here, A is a constant. Moreover, for any $\delta > 0$, we have

$$\lim_{k\to+\infty}\int_{R^n\setminus B_\delta}\left(\left|\nabla c_k^{\frac{i}{n-1}}u_k\right|^n+\left(c_k^{\frac{1}{n-1}}u_k\right)^n\right)dx=\int_{R^n\setminus B_\delta}\left(|\nabla G|^n+(G)^n\right)dx$$

$$=G(\delta)\left(t-\int_{B_\delta}G^{n-1}dx\right).$$

Proof. Slightly modifying the proof in [14], we can prove

$$G=-\frac{1}{\propto_n}\log r^n+A+o(1).$$

One can see [26] for details. Further, testing the equation (15) with 1, we get

$$\omega_{n-1}G^d(r)^{n-1}r^{n-1}=\int_{\partial B}|\nabla G|^{n-2}\frac{\partial G}{\partial n}=1-\int_{B_r}G^{n-1}dx=1+O\left(r^n\log^{n-1}r\right).$$

Then we get (16).

We have

$$\int_{R^n\setminus B_\delta}u_k^{\frac{n}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx\leq C\int_{R^n\setminus B_\delta}u_k^n\,dx\to 0\,. \qquad (17)$$

Recall that $U_k\in H_0^{1,n}\left(B_{R_k}\right)$. By equation (15) we get

$$\int_{R^n\setminus B_\delta}\left(|\nabla U_k|^n+U_k^n\right)dx=\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}\int_{R^n\setminus B_\delta}u_k^{\frac{n}{n-1}}\Phi'\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx-\int_{\partial B_\delta}\frac{\partial U_k}{\partial n}|\nabla U_k|^{n-2}U_k dS\,.$$

By (17) and (11) we then get

$$\lim_{k\to+\infty}\int_{R^n\setminus B_\delta}\left(|\nabla U_k|^n+U_k^n\right)dx=\lim_{k\to+\infty}\int_{\partial B_\delta}\frac{\partial U_k}{\partial n}|\nabla U_k|^{n-2}U_k dS$$

$$=-G(\delta)\int_{\partial B_\delta}\frac{\partial G}{\partial n}|\nabla G|^{n-2}dS$$

$$=G(\delta)\left(1-\int_{B_\delta}G^{n-1}dx\right).$$

We are now in the position to complete the proof of Theorem (1.1): We have seen in (12) that

$$\int_{R^n\setminus B_\delta}\Phi\left(\beta_k u_k^{\frac{n}{n-1}}\right)dx\leq C\,.$$

So, we only need to prove on B_R

$$\int_{B_R}e^{\beta_k u_k^{\frac{n}{n-1}}}\,dx<C\,.$$

The classical Trudinger-Moser inequality implies that

$$\int_{B_R} e^{\beta_k \left(u_k - u_k(R)^+\right)^{\frac{n}{n-1}}} dx < C = C(R).$$

By Proposition (1.14), $u_k(R) = O\left(\frac{1}{c_k^{\frac{1}{n-1}}}\right)$, and hence we have

$$u_k^{\frac{n}{n-1}} \le \left((u_k - u_k(R))^+ + u_k(R)\right)^{\frac{n}{n-1}} \le \left((u_k - u_k(R))^+\right)^{\frac{n}{n-1}} + C_1,$$

then, we get

$$\int_{B_R} e^{\beta_k u_k^{\frac{n}{n-1}}} dx < C'.$$

To proof Proposition (1.16), we will use a result of Carleson and Chang (see [12]:

Lemma (1.15) (Li and Ruf, 2000). Let B be the unit ball in R^n. Assume that u_k is a sequence in $H_0^{1,n}(B)$ with $\int_B |\nabla U_k|^n dx = 1$. If $u_k \to 0$, then

$$\limsup_{k \to +\infty} \int_B \left(e^{\propto_n |u_k|^{\frac{n}{n-1}}} - 1\right) dx \le |B| e^{1+1/2+\cdots+1/(n-1)}.$$

Then, we get the following:

Proposition (1.16) (Li and Ruf, 2000). If S cannot be attained, then

$$S > \min\left\{\frac{\propto_n^{n-1}}{(n-1)!} e^{\propto_n A + 1 + 1/2 + \cdots + 1/(n-1)}\right\}.$$

Proof. Set $u_k' = \frac{(u_k(x) - u_k(\delta))^+}{\|\nabla u_k\|_{L^n(B_\delta)}}$ which is in $H_0^{1,n}(B_\delta)$. Then by the result of Carleson and Chang, we have

$$\limsup_{k \to +\infty} \int_{B_\delta} e^{\beta_k u_k'^{\frac{n}{n-1}}} \le |B_\delta| e^{1+1/2+\cdots+1/(n-1)}.$$

By Lemma (1.15), We have

$$\int_{R^n \setminus B_\delta} \left(\left|\nabla c_k^{\frac{1}{n-1}} u_k\right|^n + \left(c_k^{\frac{1}{n-1}} u_k\right)^n\right) dx \to G(\delta)\left(1 - \int_{B_\delta} G^{n-1} dx\right),$$

and therefore, we get

$$\int_{B_\delta} |\nabla u_k|^n dx = 1 - \int_{R^n \setminus B_\delta} \left(|\nabla u_k|^n + u_k^n\right) dx - \int_{B_\delta} u_k^n dx = 1 - \frac{G(\delta) - \epsilon_k(\delta)}{c_k^{\frac{n}{n-1}}}, \quad (18)$$

where $\lim_{\delta \to 0} \lim_{k \to +\infty} \in_k = 0$.

By (12) in Lemma (1.11) we have

$$\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_\rho \setminus B_{Lr_k}} e^{\beta_k u_k'^{\frac{n}{n-1}}} = \left| B_\rho \right|,$$

for any $\rho < \delta$. Furthermore, on B_ρ we have by (18)

$$\left(u_k'\right)^{\frac{n}{n-1}} \le \frac{u_k^{\frac{n}{n-1}}}{\left(1 - \frac{G(\delta) + \in_k (\delta)}{c_k^{\frac{n}{n-1}}}\right)^{\frac{1}{n-1}}} = u_k^{\frac{n}{n-1}} \left(1 + \frac{1}{n-1} \frac{G(\delta) + \in_k (\delta)}{c_k^{\frac{n}{n-1}}} + O\left(\frac{1}{c_k^{\frac{2n}{n-1}}}\right)\right)$$

$$= u_k^{\frac{n}{n-1}} + \frac{1}{n-1} G(\delta) (\frac{u_k}{c_k}) + O\left(c_k^{\frac{-n}{n-1}}\right)$$

$$\le u_k^{\frac{n}{n-1}} - \frac{\log \delta^2}{(n-1) \propto_n}.$$

Then we have

$$\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_\rho \setminus B_{Lr_k}} e^{\beta_k u_k'^{\frac{n}{n-1}}} dx \le O\left(\delta^{-n}\right) \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_\rho \setminus B_{Lr_k}} e^{\beta_k u_k^{\frac{n}{n-1}}} dx \to \left| B_\rho \right| O\left(\delta^{-n}\right).$$

since $u_k' \to 0$ on $B_\delta \setminus B_\rho$, we get

$$\lim_{k \to +\infty} \int_{B_\delta \setminus B_\rho} \left(e^{\beta_k u_k'^{\frac{n}{n-1}}} - 1\right) dx = 0,$$

then

$$0 \le \lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_\delta \setminus B_{Lr_k}} \left(e^{\beta_k u_k'^{\frac{n}{n-1}}} - 1\right) dx \le \left| B_\rho \right| O\left(\delta^{-n}\right).$$

Letting $\rho \to 0$, we get

$$\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_\delta \setminus B_{Lr_k}} \left(e^{\beta_k u_k'^{\frac{n}{n-1}}} - 1\right) dx = 0.$$

So we have

$$\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_\delta \setminus B_{Lr_k}} \left(e^{\beta_k u_k'^{\frac{n}{n-1}}} - 1\right) dx \le e^{1 + 1/2 + \cdots + 1/(n-1)} \left| B_\delta \right|.$$

Now, we fix an L. Then for any $x \in B_{Lr}$, we have

$$\beta_k u_k^{\frac{n}{n-1}} = \beta_k (\frac{u_k}{\left\| \nabla u_k \right\|_{L^n(B_\delta)}})^{\frac{n}{n-1}} (\int_{B_\delta} \left| \nabla u_k \right|^n dx)^{\frac{1}{n-1}}$$

$$= \beta_k \big(u_k' + \frac{u_k(\delta)}{\|\nabla u_k\|_{L^n(B_\delta)}}\big)^{\frac{n}{n-1}} \big(\int_{B_\delta} |\nabla u_k|^n \, dx\big)^{\frac{1}{n-1}}$$

(using that $u_k(\delta) = O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)$ and $\|\nabla u_k\|_{L^n(B_\delta)} = 1 + O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)$)

$$= \beta_k \big(u' +_k u_k(\delta) + O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)\big)^{\frac{n}{n-1}} \big(\int_{B_\delta} |\nabla u_k|^n \, dx\big)^{\frac{1}{n-1}}$$

$$= \beta_k u_k'^{\frac{n}{n-1}} \big(1 + \frac{u_k(\delta)}{u_k'} + O\big(\frac{1}{c_k^{\frac{1}{n-1}}}\big)\big)^{\frac{n}{n-1}} \left(1 - \frac{G(\delta) + \in_k(\delta)}{c_k^{\frac{n}{n-1}}}\right)^{\frac{1}{n-1}}$$

$$= \beta_k u_k'^{\frac{n}{n-1}} \left[1 + \frac{n}{n-1}\frac{u_k(\delta)}{u_k'} - \frac{1}{n-1}\frac{G(\delta) + \in_k(\delta)}{c_k^{\frac{n}{n-1}}} + O\left(\frac{1}{c_k^{\frac{2n}{n-1}}}\right)\right].$$

It is easy to check that

$$\frac{u_k'(r_k x)}{c_k} \to 1, \quad \text{and} \quad (u_k'(r_k x))^{\frac{1}{n-1}} u_k(\delta) \to G(\delta).$$

So, we get

$$\lim_{L\to+\infty}\lim_{k\to+\infty} \int_{B_{Lr_k}} \big(e^{\beta_k u_k^{\frac{n}{n-1}}} - 1\big)dx = \lim_{L\to+\infty}\lim_{k\to+\infty} e^{\propto_n G(\delta)} \int_{B_{Lr_k}} \big(e^{\beta_k u_k'^{\frac{n}{n-1}}} - 1\big)dx$$

$$\le e^{\propto_n G(\delta)} \delta^n \frac{\omega_{n-1}}{n} \le e^{1+1/2+\cdots+1/(n-1)}$$

$$= e^{\propto_n\left(-\frac{1}{\propto_n}\log\delta^n + A + O(\delta^n \log^n \delta)\right)} \delta^n \frac{\omega_{n-1}}{n} \; e^{1+1/2+\cdots+1/(n-1)}.$$

letting $\delta \to 0$, then the above inequality together with Lemma (1.8) imply Proposition (1.16).

1. The test functions

Definition (2.1) (Li and Ruf, 2000). We will construct a function sequence $\{u_\in\} \subset H^{1,n}(R^n)$ with $\|u_\in\|_{H^{1,n}} = 1$ which satisfies

$$\int_{R^n} \Phi\Big(\propto_n |u_\in|^{\frac{n}{n-1}}\Big)dx > \frac{\omega_{n-1}}{n} e^{A+1+1/2+\cdots+1/(n-1)},$$

for $\in > 0$ sufficiently small.

Let

$$u_{\in}=\begin{cases} C-\dfrac{(n-1)\log\left(1+c_n\left|\frac{x}{\in}\right|^{\frac{n}{n-1}}\right)+A_{\in}}{\propto_n C^{\frac{1}{n-1}}} & |x|\le L_{\in} \\ \dfrac{G(|x|)}{C^{\frac{1}{n-1}}} & |x|>L_{\in}. \end{cases}$$

where $A_{\in}, C$ and L are functions of $\in$ (which will be defined later, by (19), (20), (21)) which satisfy

$(i)\, L\to+\infty, C\to+\infty$ and $L_{\in}\to 0$, as $\in\to 0$;

$$(ii)\, C-\frac{(n-1)\log\left(1+c_n L^{\frac{n}{n-1}}\right)+A_{\in}}{\propto_n C^{\frac{1}{n-1}}}$$

$$(iii)\, \frac{\log L}{C^{\frac{n}{n-1}}}\to 0 \quad as \quad \in\to 0.$$

We use the normalization of $u_{\in}$ to obtain information on $A_{\in}, C$ and L. we have

$$\int_{R^n\setminus B_{L_{\in}}}\left(|\nabla u_{\in}|^n+u_{\in}^n\right)d\,x=\frac{1}{C^{\frac{n}{n-1}}}\left(\int_{B_{L_{\in}}^{\in}}|\nabla G|^n\,dx+\int_{B_{L_{\in}}^{\in}}G^n dx\right)$$

$$=\frac{1}{C^{\frac{n}{n-1}}}\int_{\partial B_{L_{\in}}}G(L_{\in})|\nabla G|^{n-2}\frac{\partial G}{\partial n}dS$$

$$=\frac{G(L_{\in})-G(L_{\in})\int_{B_{L_{\in}}}Gdx}{C^{\frac{n}{n-1}}},$$

And

$$\int_{B_{L_{\in}}}|\nabla u_{\in}|^n\,dx=\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\int_0^{c_nL^{\frac{n}{n-1}}}\frac{u^{n-1}}{(1+u)^n}du$$

$$=\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\int_0^{c_nL^{\frac{n}{n-1}}}\frac{((1+u)-1)^{n-1}}{(1+u)^n}du$$

$$=\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\sum_{k=0}^{n-2}\frac{C_{n-1}^k(-1)^{n-1-k}}{n-1-k}$$

$$=\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\log\left(1+c_nL^{\frac{n}{n-1}}\right)+O\left(\frac{1}{L^{\frac{n}{n-1}}C^{\frac{n}{n-1}}}\right)$$

$$= -\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\left(1+1/2+1/3+\cdots+1/(n-1)\right)+\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\log\left(1+c_n L^{\frac{n}{n-1}}\right)+O\left(\frac{1}{L^{\frac{n}{n-1}}C^{\frac{n}{n-1}}}\right),$$

where we used the fact

$$-\sum_{k=0}^{n-2}\frac{C_{n-1}^{k}(-1)^{n-1-k}}{n-1-k}=1+\frac{1}{2}+\cdots+\frac{1}{n-1}.$$

It is easy to check that

$$\int_{B_{L_\in}}|\nabla u_\in|^n\,dx=O\left((L_\in)^n C^n\log L\right),$$

and thus we get

$$\int_{R^n}\left(|\nabla u_\in|^n+u_\in^n\right)dx=\frac{n-1}{\propto_n C^{\frac{n}{n-1}}}\{-(n-1)(1+1/2+1/3+\cdots+1/(n-1))+\propto_n$$

$$+(n-1)\log\left(1+c_n L^{\frac{n}{n-1}}\right)-\log(L_\in)+\phi\},$$

where

$$\phi=O\left((L_\in)^n C^n\log L+(L_\in)^n\log^n L_\in+L^{\frac{n}{n-1}}\right).$$

Setting $\int_{R^n}\left(|\nabla u_\in|^n+u_\in^n\right)dx=1$, we obtain

$$\propto_n C^{\frac{n}{n-1}}=-(n-1)(1+1/2+\cdots+1/(n-1))+\propto_n A+\log\frac{\left(1+c_n L^{\frac{n}{n-1}}\right)^{n-1}}{L^n}-\log\in^n+\phi$$

$$=-(n-1)(1+1/2+\cdots+1/(n-1))+\propto_n A+\log\frac{\omega_{n-1}}{n}-\log\in^n+\phi. \qquad (19)$$

By (ii) we have

$$\propto_n C^{\frac{n}{n-1}}-(n-1)\log\left(1+c_n L^{\frac{n}{n-1}}\right)+A_\in=\propto G(L_\in)$$

and hence

$$-(n-1)(1+1/2+\cdots+1/(n-1))+\propto_n A-\log(L_\in)^n+\phi+A_\in=\propto G(L_\in);$$

this implies that

$$A_\notin=-(n-1)(1+1/2+\cdots+1/(n-1))+\phi. \qquad (20)$$

Next, we compute $\int_{B_{L_\in}} e^{\propto_n |u|_n^{\frac{n}{n-1}}}dx$.

Clearly, $\varphi(t) = |1-t|^{\frac{n}{n-1}} + \frac{n}{n-1}t$ is increasing when $0 \le t \le 1$ and decreasing when $t \le 0$, then

$|1-t|^{\frac{n}{n-1}} \ge 1 - \frac{n}{n-1}t$, when $|t| < 1$.

Thus we have by (ii), for any $x \in B_{L_\in}$

$$\propto_n u_\in^{\frac{n}{n-1}} = \propto_n C^{\frac{n}{n-1}} \left| 1 - \frac{(n-1)\log\left(1 + c_n \left|\frac{x}{\in}\right|^{\frac{n}{n-1}}\right) + A_\in}{\propto_n C^{\frac{n}{n-1}}} \right|$$

$$\ge \propto_n C^{\frac{n}{n-1}} \left(1 - \frac{n}{n-1} \frac{(n-1)\log\left(1 + c_n \left|\frac{x}{\in}\right| + A_\in\right)}{\propto_n C^{\frac{1}{n-1}}}\right). \qquad (21)$$

Then we have

$$\int_{B_{L_\in}} e^{\propto_n |u_\in|_n^{\frac{n}{n-1}}} dx \ge \int_{B_{L_\in}} e^{\propto_n C^{\frac{n}{n-1} - n\log\left(1 + c_n \left|\frac{x}{\in}\right|^{\frac{n}{n-1}}\right) - \frac{n}{n-1}A_\in}} dx$$

$$= e^{\propto_n C^{\frac{n}{n-1}} - \frac{n}{n-1}A_\in} \int_B \frac{\in^n}{\left(1 + c_n |x|^{\frac{n}{n-1}}\right)^n} dx$$

$$= e^{\propto_n C^{\frac{n}{n-1}} - \frac{n}{n-1}A_\in} \int_0^{c_n L^{\frac{n}{n-1}}} \frac{u^{n-1}}{(1+u)^n} du$$

$$= e^{\propto_n C^{\frac{n}{n-1}} - \frac{n}{n-1}A_\in} (n-1) \int_0^{c_n L^{\frac{n}{n-1}}} \frac{((u+1)-1)^{n-2}}{(1+u)^n} du$$

$$= e^{\propto_n C^{\frac{n}{n-1}} - \frac{n}{n-1}A_\in} \in^n \left(1 + O\left(L^{-\frac{n}{n-1}}\right)\right)$$

$$= \frac{\omega_{n-1}}{n} e^{A+1+1/2+\cdots+1/(n-1)} + O\left((L_\in)^n C^n \log L + (L_\in)^n \log^n L_\in + L^{-\frac{n}{n-1}}\right).$$

Here, we used the fact

$$\sum_{k=0}^{m} \frac{(-1)^{m-k}}{m-k-1} C_m^k = \frac{1}{m+1}.$$

Then

$$\int_{B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right) dx \ge \frac{\omega_{n-1}}{n} e^{\propto_n A+1+1/2+\cdots+1/(n-1)} + O\left((L_\in)^n C^n \log L + (L_\in)^n \log^n L_\in + L^{-\frac{n}{n-1}}\right).$$

Moreover, on $R^n \setminus B_{L_\in}$ we have the estimate

$$\int_{R^n \setminus B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right)dx \geq \frac{\propto_n^{n-1}}{(n-1)!}\int_{R^n \setminus B_{L_\in}} \left|\frac{G(x)}{C^{\frac{1}{n-1}}}\right|^n dx,$$

and thus, we get

$$\int_{B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right)dx \geq \frac{\omega_{n-1}}{n} e^{\propto_n A+1+1/2+\cdots+1/(n-1)} + \frac{\propto_n^{n-1}}{(n-1)!}\int_{R^n \setminus B_{L_\in}} \left|\frac{G(x)}{C^{\frac{1}{n-1}}}\right|^n dx$$

$$+O\left((L_\in)^n C^n \log L + (L_\in)^n \log^n L_\in + L^{-\frac{n}{n-1}}\right)$$

$$= \frac{\omega_{n-1}}{n} e^{\propto_n A+1+1/2+\cdots+1/(n-1)} + \frac{\propto_n^{n-1}}{(n-1)!C^{\frac{n}{n-1}}}\left[\int_{R^n \setminus B_{L_\in}} |G(x)|^n dx\right.$$

$$\left. +O\left((L_\in)^n C^{n+\frac{n}{n-1}} \log L + \frac{C^{\frac{n}{n-1}}}{L^{\frac{n}{n-1}}} + C^{\frac{n}{n-1}}(L_\in)^n \log^n L_\in\right)\right] \quad (22)$$

We now set

$$L = -\log \in; \quad (23)$$

then $L_\in \to 0$ as $\in \to 0$. We then need to prove that there exists $C = C(\in)$ which solves equation (19). We set

$$f(t) = -\propto_n t^{\frac{n}{n-1}} - (n-1)(1+1/2+\cdots+1/(n-1)) + \propto_n A + \log\frac{\omega_{n-1}}{n} - \log \in^n + \phi,$$

since

$$f\left(\left(-\tfrac{2}{\propto_n}\log \in^n\right)^{\frac{n}{n-1}}\right) = \log \in^n + o(1) + \phi < 0$$

for $\in$ small, and

$$f\left(\left(-\tfrac{1}{2\propto_n}\log \in^n\right)^{\frac{n}{n-1}}\right) = -\frac{1}{2}\log \in^n + o(1) + \phi > 0$$

for $\in$ small, *f* has a zero in $f\left(\left(-\tfrac{1}{2\propto_n}\log \in^n\right)^{\frac{n}{n-1}}\right)$, $f\left(\left(-\tfrac{2}{\propto_n}\log \in^n\right)^{\frac{n}{n-1}}\right)$. Thus, we defined *C*, and it satisfies

$$\propto_n C^{\frac{n}{n-1}} = -\log \in^n + O(1).$$

Therefore, as $\in \to 0$, we have

$$\frac{\log L}{C^{\frac{n}{n-1}}} \to 0,$$

and then

$$(L_\in)^n C^{n+\frac{n}{n-1}} \log L + C^{\frac{n}{n-1}} L^{\frac{-n}{n-1}} + C^{\frac{n}{n-1}}(L_\in)^n \log^n L_\in \to 0.$$

Therefore, $(i),(ii),(iii)$ hold and we can conclude from (22) that for $\in > 0$ sufficiently small

$$\int_{B_{L_\in}} \Phi\left(\propto_n u_\in^{\frac{n}{n-1}}\right)dx > \frac{\omega_{n-1}}{n} e^{\propto_n A+1+1/2+\cdots+1/(n-1)}.$$

Definition (2.2) (Li and Ruf, 2000). To define the test function 2, we construct, for $n > 2$, functions $u_\in$ such that

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_\in}{\|u_\in\|_{H^{1,n}}} \right)^{\frac{n}{n-1}} \right) dx > \frac{\propto_n^{n-1}}{(n-1)!},$$

for $\in > 0$ sufficiently small.

Let $\in^n = e^{-\propto_n c^{\frac{n}{n-1}}}$, and

$$u_\in = \begin{cases} c & |x| < L_\in \\ \dfrac{-n \log \frac{x}{L}}{\propto_n c^{\frac{1}{n-1}}} & L_\in \le |x| \le L \\ 0 & L \le |x|; \end{cases}$$

where L is a function of $\in$ which will be defined later.

We have

$$\int_{R^n} |\nabla u_\in|^n = 1,$$

and

$$\int_{R^n} u_\in^n dx = \frac{\omega_{n-1}}{n} c^n (L_\in)^n + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_\in^1 r^{n-1} \log^n r \, dr.$$

Then

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_\in}{\|u_\in\|_{H^{1,n}}} \right)^{\frac{n}{n-1}} \right) dx \ge \frac{\propto_n^{n-1}}{(n-1)!} \frac{\int_{R^n} u_\in^n dx}{1+\int_{R^n} u_\in^n dx} + \frac{\propto_n^{n-1}}{n!} \frac{\int_{R^n \setminus B_{L_\in}} u_\in^{\frac{n^2}{n-1}}}{\left(1+\int_{R^n} u_\in^n dx\right)^{\frac{n}{n-1}}} dx$$

$$= \frac{\propto_n^{n-1}}{(n-1)!} - \frac{\propto_n^{n-1}}{(n-1)!} \frac{1}{1+\frac{\omega_{n-1}}{n} c^n (L_\in) + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_\in^1 r^{n-1} \log^n r \, dx}$$

$$= \frac{\propto^n}{n!} \frac{\omega_{n-1} L^n / c^{\frac{n^2}{(n-1)^2}} \int_{\in}^{1} r^{n-1} \log^{\frac{n^2}{n-1}} r}{\left(1 + \frac{\omega_{n-1}}{n} c^n \left(L_{\in}\right)^n + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_{\in}^{1} r^{n-1} \log^n r \, dx\right)^{\frac{n}{n-1}}}.$$

We now ask that L satisfies

$$\frac{c^{\frac{n}{n-1}}}{L^n} \to 0, \text{ as } \in \to 0. \qquad (24)$$

Then, for $\in > 0$ sufficiently small, we have

$$- \frac{\propto_n^{n-1}}{(n-1)!} \frac{1}{1 + \frac{\omega_{n-1}}{n} c^n \left(L_{\in}\right) + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_{\in}^{1} r^{n-1} \log^n r \, dx}$$

$$+ \frac{\propto^n}{n!} \frac{\omega_{n-1} L^n / c^{\frac{n^2}{(n-1)^2}} \int_{\in}^{1} r^{n-1} \log^{\frac{n^2}{n-1}} r}{\left(1 + \frac{\omega_{n-1}}{n} c^n \left(L_{\in}\right)^n + \frac{\omega_{n-1} n^n L^n}{\propto_n^n c^{\frac{n}{n-1}}} \int_{\in}^{1} r^{n-1} \log^n r \, dx\right)^{\frac{n}{n-1}}}$$

$$\geq B_1 L^{n - \frac{n^2}{n-1}} - B_2 \frac{c^{\frac{n}{n-1}}}{L^n}$$

$$= \left(\frac{c^{\frac{n}{n-1}}}{L^n} B_1 \frac{L^{2n - \frac{n^2}{n-1}}}{c^{\frac{n}{n-1}}} - B_2 \right),$$

where B_1, B_2 are positive constants.

When $n > 2$, we may choose $L = bc^{\frac{1}{n-2}}$; then, for b sufficiently large, we have

$$B_1 \frac{L^{\frac{n}{n-1}(n-2)}}{c^{\frac{n}{n-1}}} - B_2 = B_1 b^{\frac{n}{n-1}(n-1)} - B_2 > 0,$$

And (24) holds. Thus, we have proved that for $\in > 0$ sufficiently small

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_{\in}}{\|u_{\in}\|_{H^{1,n}}} \right)^{\frac{n}{n-1}} \right) dx > \frac{\propto_n^{n-1}}{(n-1)!}.$$

Corollary (2.3) (Shawgy and Mahgoub, 2011): Prove that for $\in > 0$

$$\int_{R^n} \Phi\left(u_{\in}\right)^{n/n-1} dx > \frac{1}{(n-1)!}.$$

Proof: For $k = \in > 0$ Results (5.1.8)(ii) implies that $\|u_\in\| < \frac{\alpha}{\beta^{n/n-1}}$. If $u_\in \to u \in H^{1,n}(R^n)$ *with* $\|u\|_{H^{1,n}(R^n)} = 1$, we have for $\alpha_n \to \alpha$ that $\beta < \alpha^{n/n-1}$. Therefore

$$\int_{R^n} \Phi\left(\propto_n \left(\frac{u_\in}{\|u_\in\|_{H^{1,n}}} \right)^{\frac{n}{n-1}} \right) dx = \int_{R^n} \Phi(\alpha\, u \in)^{n/n-1}\, d\, x > \frac{\alpha_n^{n-1}}{(n-1)!} = \frac{\alpha^{n-1}}{(n-1)!} > \frac{\beta}{(n-1)!}.$$

Hence $\int_{R^n} \Phi(u_\in)^{n/n-1}\, dx > \frac{1}{(n-1)!}$.

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On The Topological Groups and Their Compactifications

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Abstract

In this paper some topics in topological groupshas beendiscussed, and the compact spaces and compactifications of topological groups were stated.Firstly, many definitions have been statedand followed by many examples of topological groups, some theorems have been included which are propositions and lemmas as well asthe locally compact abelian topological groupswere discussed.Secondly, the paper included a compactification of topological groups and prove some theorem and some propositions concern this topic.

Keywords:Topological Groups, Compact Spaces, Abelian Topological Groups, Locally Compact, Compactifications, Group Homeomorphism.

حول الزمر الطوبولوجية وتراصها

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مُسْتَخْلَص

تم في هذه الورقة مناقشة بعض الموضوعات الخاصةبالزمر الطوبولوجية، كما شملت الفضاءات المتراصة والتراصيات في الزمر الطوبولوجية. أولاً تم إيراد العديد من التعريفات، وأُتبعت بالكثير من الأمثلة للزمر الطوبولوجية. شملت الورقة أيضاً بعض النظريات المتمثلة في المبرهنات والتمهيديات علاوة على مناقشة الزمر الطوبولوجية الابلية المتراصة محليا. ثانياً حوت الورقة التراصيات بالنسبة للزمر الطوبولوجية مع تقديم براهين لبعض النظريات والتمهيديات التي تخص هذا الموضوع

***كلمات مفتاحية:** الزمر الطبولوجية، الفضاءات المتراصة، الزمر الطوبولوجية الابلية، التراص المحلي، التراصيات، الزمر التماثلية.*

Introduction

The topology science is very important science, it has many efficient in mathematical field, and it is entering in many sciences, like physics and engineering. The important topics in topology science, are topological spaces, topological groups and compactness, which play a very important part in pure mathematics. Every group, can be made into a topological group, imposing the discrete topology on it. The topological groups have many parts, like abelian topological groups and non-abelian topological groups and topological transformation groups. The compactness and compactifications are very important in topology science, because they are referring to topological spaces and topological groups, they can be described as the maximal ideal spaces of certain functions algebras.

1. Topological groups

In this paper, we discuss some definitions and some examples of topological groups, and we state some theorems, lemmas and some propositions on topological groups. In the last section, someexplanations about locally compact abelian topological groups, with refer to [1],[3] and [5].

2.1 Definitions

The definitions below are included in [1] and [3].

Definition (2.1.1): A topological group is a triple (G,τ), where (G,) is a group and τ is a topology on G such that, the function $f: G \times G \to G$defined by$f(x, y) = x.y^{-1}$ for $x, y \in G$ is continuous.

Here $G \times G$ is viewed as a topological space by using the product topology. It is common to require that the topology on G be Hausdorff.

Definition (2.1.2): We say that (G, X,τ) is a topological group if (G, X) is a group and (G,τ) is a topological space such that, writing $M(x, y) = x \times y$ and $Jx = x^{-1}$ the multiplication map $m : G^2 \to G$ and the inversion map $J: G \to G$ are continuous.

Definition (2.1.3): Let G be a topological group, and let $a \in G$, then:

i. The map $L(a): G \to G : x \to ax$ is called a left translation.

ii. The map $R(a): G \to G : x \to xa$ is called a right translation.

Proposition (2.1.4): Let G be a topological space, which is also a group, then G is a topological group if and only if:

i. The set $\{e\}$ is closed.
ii. For all $a \in G$ the translations $R(a)$ and $L(a)$ are continuous.
iii. The mapping $G \times G \to G : (x, y) \to xy^{-1}$ is continuous at the point (e, e).

Definition (2.1.5): If (G,X_G ,τ_G) and (G,X_H ,τ_H) are topological groups, we say that, $\theta: G \to H$ is an isomorphism if it is a group isomorphism and a topological homeomorphism.

Lemma (2.1.6): let U be neighborhood of e in a topological group G, then there exists a neighborhood v of e such that $v \subset U$ and $v = v^{-1}$ and $vv = vv^{-1} \subset U$. We shall call such a neighborhood v of e symmetric.

Definition (2.1.7): A local group is a Hausdorff space N such that:

i. There is a binary operation in N , $(x, y) \to xy$ which is defined for certain pairs $(x, y) \in N \times N$.
ii. The operation is associative.
iii. There exists $e \in N$, thus for all $x \in N$, $xe = ex = x$.
iv. There exists an inverse operation in N , $X \to X^{-1} : xx^{-1} = x^{-1}x = e$.
v. The maps $(x, y) \to xy$ and $x \to x^{-1}$ are continuous.

Definition (2.1.8):

i. The local groups N and N' are topologically isomorphic if there exists a homeomorphism $f: N \to N' : x \to f(x)$, such that the product xy is defined in N if and only if the product $f(x)f(y)$ is defined in N', andin this case $f(xy) = f(x)f(y)$.
ii. The topological groups G and G' are locally isomorphic, they have open nuclei which is local group, are topologically isomorphic.

Definition (2.1.9): Topological space M is a $T_0 - space$, if for any given pair of distinct points $x, y \in M$, there exists an open set U of M, which contains one of the points, but not the other.

Definition (2.1.10): A $T_0 -$ topological group is a group G which is $T_0 - space$, and such that, the map, $G \times G \to G : (x, y) \to xy^{-1}$, is continuous.

Definition (2.1.11): A topological space M is a $T_1 - space$ if for distinct points $x \neq y \ in \ M$, there exists an open set $V \ with \ y \in V \ but \ x \in V$.

Definition (2.1.12): Let M a Hausdorff topological space, and let G be a topological group, then:

i. G operates on M if there is, a surjection $G \times M \to M : (g, p) \to g.p$ Such that $(g_1 g_2).p = g_1.(g_2.p)$, and $e \cdot p = p$ for all $g_1, g_2 \in G$ and $p \in M$ where e is identity of G.

ii. G operates transitively on M, if for every $p, q \in M$, there exists $g \in G$ Such that $g \cdot p = q$.

iii. G operates continuously on M, if the map $G \times M \to M : (g, p) \to g.p$ is continuous.

iv. G is called a topological transformation group on M, if G operates Continuously on M.

v. G is effective if $a \cdot p = p$, for all $p \in M$ emplies $a = e$.

vi. Let p be fixed in M, then $G(p) = \{g \in G : g \cdot p = p\}$ is a group called the isotropy subgroup of G at p. The set $G \cdot p = \{g \in G : g \cdot p = p\}$, is called an orbit under G.

Definition (2.1.13): A homogenous space M is a space with a transitive group action by Lie group action implies that there is only one group orbit, M is isomorphic to quotient space G/H, where H is the isotropy group G.

2.2 Examples

Example (2.2.1): Examples of abelian topological groups

Before we state the examples, we assert that; every group can made into a topological imposing the discrete topology on it. Here are some important examples of an abelian topological groups:

1. All Euclidean spaces under the usual additional, are abelian topological groups.

2. The non-zero real numbers, or the non-zero complex numbers under multipli-cation form an abelian topological group.
3. All topological vector spaces, such as Banach spaces, or Hilbert spaces are abelian topological groups.
4. Let X be a space and $H(X)$, is the group of all homeomorphisms of X . If $X$is locally compact and regular, then $H(X)$, becomes an abelian topological group under composition.
5. If $\{G_i : i \in I\}$ is a collection of topological groups, the product space $\prod_{i\in I} G_i$ can be made into topological group under co-ordinate wise multiplication.
6. Let H a subgroup of a topological group G and $G/H$is the space of right cosets of$H$ in G then, if H is a normal subgroup of G , then G/H is a topological group.

Examples (2.2.2): of non-abelian topological groups

All examples and properties below are in (Arther and Ralph, 1973).

Consider the subgroup of rotations of R^3, generated by two rotations by irrational of multiplies of 2π about different axes.

All the above examples are Lie groups (topological groups that are also manifolds).

An example of a topological group which is not Lie group is given by the rational numbers Q ,this a countable space, and it does not have the discrete topology.

2.3 Properties:

1 If a is an element of a topological group G,then left or right multiplication with $a$yields a homeomorphism$G \to G$, thus can be usedto show that all topological groups are actually uniform spaces. Every topological group can be viewed as a uniform space in two ways, the left uniformly turns all left multiplication into uniformly continuous maps, while the right uniformly turns all right multiplication into uniformly continuous maps. If G is not abelian, these two need not coincide.

2 As a uniform space, every topological group is completely regular. It follows that if a topological group is $T_0(i.e. Kolmogorov)$,then it is already T_2 (i.e. *Hausdorff*).

3 The most natural notion of homeomorphism between topological groups is that of a continuous group homeomorphism. Topological groups, together with continuous group homeomorphisms as morphemes, form a category.

4 If H is a normal subgroup of the topological group G, then, the factor group G/H becomes a topological group by using the quotient topology (the finest topology on G/H which makes the natural projection $G \to G/H$ continuous).

5 The algebraic and topological structures of a topological group interact in non-trivial ways, for example, in any topological group the connected component containing the identity element, is a normal subgroup.

2.4 Subgroups and nuclei

Definition (2.4.1): Let G be a topological group, and let H be a subset of G such that, $HH^{-1} \subset H$, then H is a subgroup of G .(Arther and Ralph, 1973)

Corollary (2.4.2): Let H be a closed normal subgroup of the topological group G and G/H is a quotient, then G/H becomes a topological group such that the projection $\pi: G \to G/H$ is an open continuous homeomorphism. (Arther and Ralph, 1973)

Definition (2.4.3) (Arther and Ralph, 1973**):** Let G be a topological group, then the center C of G equals $\{x \in G : xa = ax \ \forall \, a \in G\}$. The center is a normal subgroup of G,and is also denoted by $Z(G)$. (Arther and Ralph, 1973)

Definition (2.4.4): Discrete subgroups of Euclidean spaces under usual additional are known as Lattices. (see [1])

Definition (2.4.5): Let (G, X, T) be a topological group, and H a subgroup of *G*: (Torner, 2003)

i. The topological closure $\overline{H}$ of H is a subgroup.
ii. If H is normal, so is $\overline{H}$.
iii. If H contains an open set, then H is open.
iv. If H is open, then $\overline{H}$ is closed.
v. If H is closed and of finite index in G , then $\overline{H}$ is open.

Lemma (2.4.6) : Let (G, X, T) be a Hausdorff topological group, if H is a closed normal subgroup, then G/H is Hausdorff. (see [1])

Lemma (2.4.7): Let (G, X, T) be a topological group, then $I = \{\overline{e}\}$ is a closed normal subgroup. (see [1])

Definition (2.4.8) : If G is a topological group, the connected component of identity $e \in G$ is called the identity component of G, and is denoted by G_0 , and $G_0$is closed normal subgroup of G , and the connected component $c(a)$ of $a \in G$ equals aG_0. (Arther and Ralph, 1973)

Definition (2.4.9): A subset of atopological group$G$,which contains an open neighborhood of the identity e, is called a nucleus of G. (Arther and Ralph, 1973)

Proposition (2.4.10): let V be the family of all nucleus of a topological group G, then V satisfies: (see [1])

i. $v_1 , v_2 \in V$ implies $v_1 \cap v_1 \in V$;
ii. $v_1 \in V^2$ and $v_1 \subset W \subset G$ implies $W \in V$;
iii. For any $v_1 \in V$, there exists $v^{-1} \in V$ such that $vv^{-1} \in v_1$;
iv. If $v \in V$ and $a \in G$, then $v^{-1} \in V$;
v. $v^{-1} \cap \{v: v \in V\} = \{e\}$.

2.5 Locally compact abelian topological groups

Let A be an abelian group, thus A is a set equipped with a binary operation $+$, which is commutative and associative, there is identity element $0 \in A$, such that $0 + a = a$ for all $a \in A$, and each $a \in A$ has an inverse $-a$, characterized by $a + -a = 0$, as a basic examples, the integers, real numbers and complex numbers are abelian groups under addition, and for each positive number n we have the integers modulo n , a cyclic group with n elements.

Let us also assume that A is a topological space, which is to say that certain subsets of A are designated as open subsets. As usual one required that the empty set and A itself are open subsets of A , that the intersection of any finite collection of open subsets is an open subset, and that the union of any collection of open sets, is an open subset. Once the open subsets are selected, the closed subsets are defined to be the complements in A of the open subsets.

Various standard notions, such as continuity at a point of a mapping between two topological spaces, can be defined in terms of the open subsets through standard methods.

To say that A is a topological group, means that the group structure and topo-logy are compactible in a natural way. Specifically, the group operation $+$ should be continuous as a mapping from $A \times A$ into A, and $a \rightarrow -a$, should be continuous as a mapping from A to it self. This implicitly uses the product topology on$A \times A$ is open if it is the union of products of open subsets of A. It is customary to require that A be a Hausdorff topological space, which is equivalent in this setting to the requirement that $\{0\}$ be a closed subset of A.

In any topological space a subset K is said to be compact if every open covering of K in the space admits a finite sub covering, i.e., if for every family$\{U_i\}_{i \in I}$ of open subsets of the topological space such that:

$(i) K \subseteq U_{i \in I} u_i$.

There is a finite collection $i_1, \cdots, i_I$of indices in I such that:

$(i) K \subseteq U_{iI} u, \cdots u U_{iI}$.

A topological space is said to be locally compact if for each point x in the space, there is an open subset W and a compact subset K of the space such that:

$$(i) x \in W \subseteq K.$$

A locally compact abelian topological group is an abelian topological group which is locally compact as a topological space.

Of course, local compactness at the identity element e implies local compact-ness at every point because group translations define homeomorphism.

As in (3), it is simpler to say “LCA group” in place of locally compact abelian topological group. The integers modulo$n$ are natural examples of LCA groupequipped with their discrete topologies, in which every subset is considered to be open. For the real and complex numbers, one can use their standard topologies, indeed by the usual Euclidean metrics, to get LCA group. One can also consider the nonzero complex numbers using multiplication as the group operation and the usual topology. If one takes the complex numbers with modulus 1 using multiplication as thegroup operation and the usual topology, one gets acompact LCA group.

Fix an integer $n \geq 2$, and consider the group consisting of sequences $x = \{x_j\}_{j-1}^{\infty}$ such that each of them is an integer such that $0 \leq x_j \leq n-1$, and the sum of two elements x, y in the group A is defined by adding each term modulo n. If x is an element of the group and$i$ is a positive integer, then the ith standard neighborhood around$x$ is defined to be the set of y in the group such that $x_j = y_j$, then $1 < j < I$. This leads to a topology on the space in which a sub set of the group is open, if for each point x which is contained in the subset. Well known results in topology imply that this space is compact with respect to this topology, and in fact it is homeomorphic to the cantor set.

It is easy to see that this example defines a compact LCA group. Namely, the group operations are continuous with respect to the topology just defined. This is a nice example where the topological dimension is equal to e ,which isto say that the space is totally disconnected, with no connected subsets with at least two elements. As the same time the topology is not the discrete topology.

Let A be a LCA group. A basic object of interest associated to A is translation-invariant integral, which is a linear mapping from the vector space of complex-valued continuous functions as $f(x)$ on A with compact support into the complex number such that the integral of $f(x+a)$ is equal to the integral of $f(x)$ for all a ∈ A , the integral of a real valued function is a real number, the integral of a non-negative real number, and the integral of f is positive, if f is a non-negative real valued continuous function on A with compact support such that $f(x) > 0$ for some $x \in A$. In the examples described earlier such that an integral can be defined explicitly, in terms of sums, classical Riemann integrals, or simple generalizations of Riemann integrals for the spaces of sequences modulo n . A general theorem states that any LCA group$A$ an invariant integral, and that this integral is unique except for multiplying it by a positive real number.

Let A be an abelian group. By a character on A we mean a continuous group homeomorphism from A into the group of complex numbers with modulus I with respect to multiplication. Sometimes one may wish to consider unbounded characters more generally, which are continuous homeomorphisms from$A$ into the non-zero complex numbers with respect to multiplication. Note that any bounded subgroup of the nonzero complex numbers with respect to multiplication is contained in the complex numbers with modulus I as one can easily verify. Thus, a bounded, continuous homeomorphism from A into the group of nonzero complex numbers is a character, and in particular every continuous homeomorphism from A into nonzero complex numbers is a character when A is compact.

If $f(x)$is a complex valued continuous function on A with compact support, or an integrable function more generally, one can define its Fourier transformation $\hat{f}(\emptyset)$by saying that if $\emptyset$ is a character on A , then $\hat{f}(\emptyset)$is the integral of f times the complex numbers conjugate of $\emptyset$, using a fixed invariant integral on A as discussed previously. If A is not compact, then one can extend this to a Fourier – Laplace transform by allowing unbounded characters, at least when f has compact support or sufficient integrality properties. For bounded character one has the usual inequality which states that $\left|\hat{f}(\emptyset)\right|$ is less than or equal to the integral of $|f|$.

Many classical a sects of Fourier analysis work in this setting. A basis point is that the Fourier transform diagonalizes translation operator, which means that if $a \in A$ and $f(x)$ is continuous function on the group with compact support, or an integrable function on the group, then the Fourier transform of $f(x-a)$at the character $\emptyset$ is equal to $\overline{\emptyset(a)}$ times as Fourier transform of $f(x)$at $\emptyset$. One can also define convolution in the usual way, $\bigoplus$using the invariant integral on A , and the Fourier transform of a convolution is equal to the product of the corresponding Fourier transforms.

3 Topologicalgroups' compactificationsIntroduction

Every topological group G has some natural compactifications. They can bedescribed as the maximal ideal spaces of certain functions algebras, or as the Samuel compactifications for certain uniformities on G , some compactifications of G carry an algebraic structure, and may be useful for studying the group G itself.

We consider, in particular, the following constructions: the greatest ambit $S(G)$ and the universal minimal compact$G$-space M_G(sections 2 and 3); e Roelcke compactifications $R(G)$ (section 4) ; the weakly almost periodic compactifications $W(G)$(section 5). In the last case the canonical map $G \rightarrow W(G)$ need not be an embedding. In section 6 , we discuss the group of isometrics of the Urysohn universal metric space u, all these topics are included in [4] and [5].

3.2 Greatest ambit $S(G)$

Let G be a topological group, the Banach space $B = RUC^b(G)$ of all right uniformly continuous bounded complex functions on G is a $C^* - algebra$, and G acts on B by $C^* - algebra$ automorphisms. Let $S(G)$be the compact maximal ideal space of B . It is the least compactifications of G over which all functions from B can extended. The topological group of all $C^* - algebra$

automorphisms of$B$ is naturally isomorphic to$H\big(S(G)\big)$. It follows that G acts on $S(G)$, and the natural homeomorphism$G \to H(S(G))$ is a topological embedding.

The space $S(G)$ can also be described as the Samuel compactifications of the uniform space (G, R) , here R is the right uniformity on G . The basic entourages for R are of the form $\{(x, y) \in G : xy^{-1} \in V\}$, where $V \in N(G)$. The Samuel compactifications of a uniform space (x, u) is the completion of x with respect to the finest pre-compact uniformity which coarser than u .

We shall consider G as dense subspace of $S(G)$. The action $G \times S(G) \to S(G)$ extends the multiplication $G \times G \to G$.

A $G - space$ is a topological space X with a continuous action of G , that is, a map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $ix = x$, $(g, h \in G, x \in X)$. A $G - map$ is a map $f : X \to Y$ between $G - spaces$ such that$f(gx) = gf(x)$ for all $x \in X$, $g \in G$. The $G - spacesS(G)$ has distinguished point e (the unity), and the pair $(S(G), e)$ has the following universal property : for every compact $G - space$ X and every $p \in X$ there exists a unique $G -$ map $f: S(G) \to X$ such that $f(e) = p$. Indeed, the map$g \to gp$ from G to X is $R -$ uniformly continuous and hence can be extended over $S(G)$.

Theorem (3.2.1) : For every topological group G the greatest ambit $X = S(G)$ has a natural structure of a left topological semi group with a unity such that the multiplication $X \times X \to X$ extends the action $G \times X \to X$.

Proof

Let $x, y \in X$ such that $r_y(e) = y$. Define $xy = r_y(x)$. Let us verify that the multiplication $(x, y) \to xy$ has the required properties. For a fixed y the map $x \to xy$ is equal to ry and hence is continuous. If $yz \in X$, the self-maps$r_z r_y$ and r_{zy} of X are equal, since both are $G -$ maps sending e to $yz = r_z(y)$. This means the multiplication on nX is associative. The distinguished element $e \in X$ is the unity of X : we have $= r_x(x) = x$. If $g \in G$ and $x \in X$, the expression gx can be understood in two ways: in the sense of exterior action of G on X as a product in X.To see that these two meanings agree, note that $r_x(g) = r_x(ge) = gr_x(e) = gx$ (the exterior action is meant in the last two terms; the equality holds since r_x is a $G - map$.

3.3 Universal minimal compact $G - space$

Definition (3.3.1):A $G - space$ is minimal if it has no proper$G -$ invariant closed subset or, equivalently, if the orbit Gx is dense in X for every $x \in X$.

Lemma (3.3.2) : The Universal minimal compact $G - space$ M_G is characterized by the following property: M_G is minimal compact $G - space$ and for every compact minimal $G - space\ X$ there exists a $G - map$ of M_G onto X.

Since Zorn′s lemma implies that every compact $G - space$ has a minimal compact $G - subspace$, it follows that for every compact $G - space\ X$, minimal or not, there exists a $G - map$ of M_G to X.

Lemma (3.3.3): The existence **of** M_G is easy: take for M_G any minimal closed $G - subspace$ of $S(G)$ universal property of $(S(G), e)$ implies the correspond-ding universal property of M_G . It is also true that M_G is unique, in the sense that any two Universal minimal compacts$G - spaces$ are isomorphic.

Proposition (3.3.4): If $f: X \to X$ is a $G - self - map$and $a = f(e)$then $f = r_a$.

Proof: We have $f(x) = f(xe) = x\big(f(e)\big) = xa = r_a(x)$ and hence for all $x \in X$.

A subset $I \subset X$ is a left ideal if $XI \subset I$. Closed $G -$ subspaces of X are the same as closed left ideals of$X$. An element$x$ of asemi group is an idempotent if $x^2 = x$. Every closed $G -$subspace of X , being aleft ideal, is moreover a left topological compact and hence contains an idempotent.

Theorem (3.3.5): Every non-empty compact left topological semi group K contains an idempotent.

Proof: Zorn′s lemma implies that there exists a minimal element Y in the set of all closed non-empty sub semi groups of K. Fix $a \in Y$. We claim that $a^2 = a$(and hence Y is a sigleton). The set Ya, being aclosed semi group of Y. It follows that the closed sub semi group $Z = \{x \in Y : xa = a\}$ is non-empty. Hence $Z = Y$ and $xa = a$ for every $x \in Y$. It particular, $a^2 = a$.

Let M be a minimal closed left ideal of X. We have just proved that there an idempotent $P \in M$. Since XP is closed left ideal contained in M, we have $XP = M$. It follows that $xp = x$ for every $x \in M$. The $G - map$ $r_p : X \to M$ defined by $R_P(X) = XP$ is a retraction of x onto M.

Proposition (3.3.6): Every $G - map f : M \to M$ has the form $f(x) = xy$ for some $y \in M$.

Proof: The composition $h = fr_P: X \to M$ is a $G - map$ of X into itself, hence it has the form $h = r_y$, where $y = h(e) \in M$ (Proposition 3.3.1). Since $r_P \uparrow M = Id$, we have $f = h \uparrow M = r_y \uparrow M$.

Proposition (3.3.7): Every $G - map f : M \to M$ is bijective.

Proof: According to Proposition (3.3.6), there is $a \in M$ such that $f(x) = xa$ for all $x \in M$. Since Ma is a closed left ideal of X contained in M, we have $Ma = M$ by the minimality of M. Thus there exists $b \in M$ such that $ba = p$.Let every $g : M \to M$ be the $G -$ map defined by $g(x) = xb$. Then $fg(x) = xba = xb = x$ for every $x \in M$, and therefore $fg = I$ (the identity map of M). We have proved that in the semi group S of all $G - self - map$ of M, every element has a right inverse. Hence S is a group (alternatively, we first deduce from the equality f g=*1* that all elements of S are surjective and then, applying this to g , we see that f is also injective.)

Theorem (3.3.8): For every topological group G the action of G on the universal minimal compact $G - space$ M_G is not 3-transitive.

For example, if K is a compact manifold of dimension > 1, or a compact Menger manifold and $G = H(K)$,then $M(G) \neq K$, since the action of G on K is 3-transitive.

It would be interesting to understand what is $M(G)$ in this case.

Let P be the pseudo arc (= the unique hereditarily indecomposable chainable continuum) and $G = H(P)$. The action of G on P is transitive but not 2-transitive, and the following question remain open:

Let P the pseudo arc and $G = H(P)$. Can M_G be identified with P ?

Question (3.3.9): Let G be abelian topological group. Suppose that G has no non-trivial continuous characters $X : G \to T$. Is G extremely amenable.

For cyclic group the question can be reformulated as follows: Let K be a compact space, and let $f \in H(K)$ be a fixed-point free homeomorphism of K . Let G be the cyclic subgroup of $H(K)$ generated by f . Does there exists a complex number a such that $|a| = I,\ a \neq I$, and the homeomorphism $X : G \to T$ defined by $X(f^n) = a^n$ is continuous.

If K is a circle, the answer is yes: for every orientation-preserving homeomorphism$f$ of a circle, the rotation number is defined which gives rise to a non-trivial continuous character on the group generated by f .

A positive answer to question (3.3.9) would imply the solution of the problem: Is it true that for every big set S of integers, the set $S - S$ contains a neighborhood of zero for Bohr topology on Z ? A set S of integers is said to be big (or syndetic) if $S + F = Z$ for some finite $F \subset Z$; this means that the gaps between consecutive terms of$S$ are uniformly bounded. The Bohr topology on$Z$ generated by all characters $X : Z \to T$. It is known that for every big subset $S \subset Z$ the $\mathrm{S - S +}$ S contains a Bohr neighborhood of zero.

Extremely amenable groups can be characterized in terms of big sets. A subset S of a topological group G is big on the left, or left syndetic, if $FS = G$ for some finite $F \subset G$.

Theorem (3.3.10): A topological group$G$is extremely amenable if and only if whenever $S \subset G$ is big on the left, SS^{-1} is dense in G.

Theorem (3.3.11): A topological group G is extremely amenable if and only if for every bounded left uniformly continuous function f from G to a finite dimensional Euclidean space, every $\varepsilon > 0$, and every finite (orcompact)$K \subset G$ there exists $g \in G$ such that diameter $f(gk) < \varepsilon$.

3.4 Roelcke Compactifications

Definition (3.4.1): For a topological group$G$let $R(G)$ be the maximal ideal of the $C^* -$ algebra of all bounded complex functions on G which are both left and right uniformly continuous. The space $R(G)$ is the Samuel compactifications of theuniform space$(G, \ell \wedge R)$, where ℓ is the left uniformly on G, R is the rightuniformly, and $\ell \wedge R$ is the Roelcke uniformly on G , the greatest lower bound of ℓ and R . We call $R(G)$ Roelcke compactifications of G .

Lemma (3.4.2): While the greatest lower bound of two compactible uniformities on a topological space in general need not be compatible, the Roelcke uniformity is compatible with the topology of G . The covers of the form $\{UxU : x \in G, U \in N(G)\}$, constitute a base of uniform covers of Roelcke uniformity.

If G is abelian, $(G) = S(G)$. In general,$R(G)$ is a $-space$, and the identity map of G extends to a $G - map\ S(G) \to R(G)$.

Definition (3.4.3): The group G is precompact if one of the following equivalent properties holds:

i. (G, L) is precompact.
ii. (G, R) is precompact.
iii. G is a subgroup of a compact group.

It can be shown that G is precompact if and only if G for everyneighborhood U of unity, there exists a finite $F \subset G$ such that $G = F \cup F$. Let us say that G is Roelcke precompact if the Roelcke uniformity $\ell \wedge R$ is precompact. This exists a finite $F \subset G$ such that $G = F \cup F$. There are many non-abelian non-precompact groups which are Roelcke compactifications. For example, the symmetric group $symm\ (E)$ of all permutations of a discrete space E , or the unitary group $U(H)$, on a Hilbert space H , equipped with the strong operator topology, are Roelcke precompact. The Roelcke compactifications of these groups can be explicitly described with the aid of the following construction:

Suppose that G acts on a compact space K . For $g \in G$, let $r(g) \subset K^2$ be the graph of the $g -$ shift $x \to gx$. The map $g \to \Gamma(g)$ from G to $Exp\ K^2$ is both left and right uniform continuous (if the compact space $Exp\ K^2$ is equipped with its unique compactible uniformity), hence it extends to a map $f_k : R(G) \to Exp\ K^2$. If the action of G on K is topologically faithful, the map f_k often happens to be an embedding in which case $R(G)$ can be identified with the closure of the set $\{\Gamma(g): g \in G\}$ in $Exp\ K^2$. For example, this the case if $K = S(G)$ or $K = R(G)$.

The space $Exp\ K^2$ is the space of all closed relations on K. It has a rich structure, since relations can be composed, reversed, or compared by induction. This structure is party inherited by $R(G)$. Let us consider some examples.

Example (3.4.4): Let $G = symm(E)$ be the topological symmetric group**.** It acts on the compact cube$K = 2^E$ **.** The natural map$f_k : R(G) \to Exp\ K^2$in an embedding.

Example (3.4.5): Let Gbe the unitary group$U(H)$, of a Hilbert space$H$, equipped with the strong operator topology (this is the topology of point wise convergence inherited from the product H^H). Let K be the unit ball of H . Equip K with the weak topology. Then K is compact. The unitary group G acts on K , and the map $R(G) \to Exp\ K^2$ is an embedding.

The space $R(G)$ has a better description in this case : $R(G)$ can be identified with the unit ball θ in Banach algebra $B(H)$ of all bounded linear operators on H. The topology on θ is the weak operator topology: the map $A \to A/K$ which assigns to every operator of norm ≤ 1 its restriction to K is a homeomorphic embedding of θ into the compact space K^K. Thus $R(G)$ has a natural structure of semi topological semi group.

Example (3.4.6): Let K be a zero-dimensional compact space such that all non-empty clopen subsets of K are homeomorphic to K . Let $G = H(K)$, the natural map $f_k : R(G) \to Exp\ K^2$ is an embedding. Moreover, the image of f_k , which is the closure of the set of all graphs of self-homeomorphisms of K , is the set θ of all closed relations on K whose domain and range are equal to K . Thus $R(G)$ can be identified with θ .

This time $R(G)$ is an ordered semi group, but not a semi topological semi group, since the composition of relations is not a separately continuous operation. As in the pervious example, one can use the space $R(G)$ to prove that G is minimal. Moreover, every non-constant onto group homeomorphism $f : G \to H$ is an isomorphism of topological groups. To prove this, we proceed as before extend f to $f : G \to H$ and look at the kernel $S = F^{-1}(e_n)$. Zorn's lemma implies the existence of maximal idempotent in S(with respect to the inclusion). Symmetric idempotent above the unity (= the identity relation = the diagonal of K^2) in $\theta$are precisely closed equivalence relations on K . Since there are no non-trivial choices for S either $S = \{1\}$ or $S = \theta$.

Example (3.4.7): Let $G = H_+(1)$be the group of all orientation-preserving homeomorphisms of the closed interval $I = [0,1]$. The map $f_G : R(G) \to Exp\ I^2$ is a homeomorphic embedding. Thus $R(G)$ can be identified with the closure of the set of all graphs of strictly increasing functions $h : I \to I$ such $h(0) = 0$ and $h(1) = 1$.

This closure consists of all curves $C \subset I^2$ which lead from (0,0) to (1,1) and like graphs of increasing functions, with the exception that$C$ may include both horizontal and vertical segments.

There seems to be no natural semi group structure on $R(G)$. This observation leads to an important result: The group G has no non-trivial homeomorphisms to compact semi topological semi groups and has no non-trivial representation by isometries in reflective Banach space.

3.5 WAP compactifications

Definitions (3.5.1): Let S a semi group and a topological space. if the multiplication $(x, y) \to xy$ is separately continuous (this means that the maps $x \to ax$ and $x \to xa$ are continuous for every $a \in S$), we say that S is a semi topological semi group.

For a topological group G let $f : G \to W(G)$ be the universal object in the category of continuous semi group homeomorphisms of G to compact semi-topological semi groups. In other words, $W(G)$ is a compact semi topological semi group, and for every continuous homeomorphism $g : G \to S$ to a compact semi topological semi group$S$ there exists a unique homeomorphism $h: W(G) \to S$ such that $g = hf$.

The existence of$W(G)$follows from twofacts: (1) arbitrary products are defined in the category of compact semi topological semi groups; (2) the cardinality of a compact space has an upper bound in terms of its density. The space $W(G)$can also be defined in terms of weakly almost periodic functions. Recall the definition of such functions.

Let a topological group G act on a space X . Denote by $C^b(X)$ the Banach space of all bounded complex valued continuous functions on Xequipped with the supremum norm. A function $f \in$

$C^b(X)$ is called weakly almost periodic ($w.a.p.$ for short), if the G-orbit of f is weakly relatively compact in the Banach space $C^b(X)$.

In particular, considering the left and right actions of a group G on it self, we can define left and right weakly almost periodic functions on G . These two notions are actually equivalent, so we can simply speak about $w.a.p.$ functions on a group G. The space WAP of all $w.a.p.$ functions on a group G is a $C^* -$ algebra , and the maximal ideal space of this algebra can be identified with $W(G)$.Thus the algebra WAP is isomorphic to the algebra $C(W(G))$of continuous functions on $W(G)$. We call $W(G)$ the weakly almost periodic $w.a.p.$ compacti-fications of the topological group G .

Remark (3.5.2): We show a compactification of a topological space X, we have a compact Hausdorff space K together with a continuous map $j : X \to K$

with a dense range. We do not require that j be a homeomorphic embedding.

For every reflexive Banach space X , there a compact semi topological semi group $\theta(X)$ associated with X : the semi group of all linear operators $A : X \to X$ of norm ≤ 1 , equipped with the weak operator topology. Recall that a Banach space X is reflexive if and only if the unit ball B in X is weakly compact. If X is reflexive, $\theta(X)$ is homeomorphic to a closed sub space of B^B(where B carries the weak topology, and hence compact) .

It turns out that every compact semi topological semi group embeds into $\theta(X)$ for some reflexive X .

Theorem (3.5.3): Every compact semi-topological semi-group is isomorphic to a closed sub semi group of , $\theta(X)$ for some reflexive Banach space.

The group of invertible elements of $\theta(X)$ is the group $Is_w(X)$of isometries of X, equipped with the weak operator topology. This topology actually coincides with the strong operator topology.

Theorem (3.5.4): For every reflexive Banach space, the weak and strong operator topologies on the group $Is(X)$ agree.

In particular, the group of invertible elements of $\theta(X)$ is a topological group. The natural action of this group on $\theta(X)$is (jointly) continuous. This can be easily deduced from the fact that the topological group $Is_S(X^*) = Is_w(X^*)$ are canonically isomorphic. In virtue of Theorem (3.5.3), similar assertions hold true for every compact semi topological semi group S : the group$G$ of invertible elements of$S$is atopological group, and the map$(x, y) \to xy$ is joint continuous on $G \times S$. Thus S is a G-space.

It follows for every topological group G the compact semi topological semi group $W(G)$ is a G-space , hence there exists a G-map $S(G) \to W(G)$ extending the canonical map $G \to W(G)$In terms of function algebras, this means that every $w.a.p.$ function on G is right uniformly continuous , since the algebra WAP is invariant under the inversion on G , $w.a.p.$ functions are also left uniformly continuous and hence Roelcke uniformly continuous .

It follows that there is a natural map $R(G) \to W(G)$. If $G = U(H)$ is the unitary group of a Hilbert space H , then $R(G) = \theta(H)$ is a compact semi topological semigroup,and therefore the canonical map $R(G) \to W(G)$ is a homeomorphism, thus $W(G) = \theta(H)$ .The canonical map$S(G) \to W(G)$ is a homeomorphism if and only if G is precompact.

In virtue of Theorem (3.5.3) and a (3.5.4), the following two properties are equivalent for every topological group G :

1. The canonical map $G \to W(G)$ is injective.
2. There exists a faithful representation of G by isometries of reflexive Banach space.

Similarly, the canonical map $G \to W(G)$ is homeomorphic embedding if and only if G is isomorphic to a topological sub group of $Is(X)$ for some reflexive Banach space.

Theorem (3.5.5): Let $G = H_+(I)$ be the group of all orientation preserving homeomorphisms of $I = [0,1]$. Then W(G) is a singleton. Equivalently, every $w.a.p.$ function on G is constant.

3.6 The group$\boldsymbol{Is(U)}$

Definition (3.6.1): The group's of Urysohn universal metric space U .

A metric space M is w-homogeneous if every isometry between two finite subsets of M extends to an isometry of$M$into itself. A metric space M is finitely injective f it has the following property: If K a finite metric space and $L \subset K$, then every isometric embedding $K \rightarrow M$. The Urysohn universal space U is the unique complete separable metric space with the following properties: (1) U contains an isometric copy of any separable metric space; (2) U is w -homogenous. Equivalently, U is the unique finitely-injective complete separable metric space. The uniqueness of U is easy: Given two separable finitely injective spaces U_1 and U_2 , one can use the "back and forth" or "shuttle" method to construct an isometry between countable dense subsets of U_1 and U_2 . If U_1 and U_2 are complete, they are isometric themselves.

Let $G = Is(U)$. The group G is a universal topological group with a countable base; every topological group H with a countable base is isometric to a subgroup of G . The idea of the proof is first to embed G into $Is(M)$ for separable metric space M and then to embed M into U in such a way that every isometry of M has a natural extension to an isometry of U .

Let (X, d) be a metric space. We say that a function $f : X \rightarrow R_+$ is Katetov if: $|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$ for all $x, y \in X$. A function f is katetov if and only if there exists a metric space $Y = X \cup \{p\}$ containing X as a subspace such that $f(x)$ for every $x \in X$ is equal to the distance between x and p . Let $E(X)$ be the set of all Katetov functions on X , equipped with the sub-metric. If Y is a non-empty subset of X and $f \in E(Y)$, define $g = K_Y(f) \in E(X)$ by

$$g(x) = inf\{d(x, y) + f(y): y \in Y\}$$

for every $x \in X$. It is easy to check that g is indeed a Katetov function on X and that g extends f . The map $K_r : E(Y) \rightarrow E(X)$ is an isometric embedding. Let $X^* = U\{K_r(E(Y): Y \subset X, Y\ )\ is\ finite\ and\ non-empty\} \subset E(X)$.

For every $x \in X$ let $h_x \in E(X)$ be the function on X defined by $h_x(y) = d(x, y)$. Note that $h_x = K_{\{x\}}(0)$ and hence $h_x \in X^*$. The map $x \rightarrow h_x$ is an isometric embedding of X intoX^* Thus we can identify X with a subspace of X^* . If K is a finite metric space, $L \subset$

$K\ and\ |K/L| = 1$, then every isometric embedding of L into X can be extended to an isometric embedding of L into X^* .

Every isometry of X has a canonical extension to isometry of X^* , and we get an embedding of topological group $Is(X) \rightarrow Is(X^*)$. (Note that the natural homeomorphism $Is(X) \rightarrow Is(E(X))$ in general need not be continuous). Iterating the construction of X^* , we get an increasing sequence of metric spaces $X \subset X^* \subset X^{**} \cdots \cdots$. Let Y be the union of this sequence, and let $\bar{Y}$ be the completion of Y . We have a sequence of embedding of topological group

$$Is(X) \rightarrow Is(X^*) \rightarrow E(X^{**}) \rightarrow E(Y) \rightarrow E(\bar{Y})$$

The space Y is finitely-injective. $\bar{Y}$ the completion of a finitely-injective. Assume that X is separable, then Y is separable, and $\bar{Y}$ is a complete separable finitely-injective metric space. Thus $\bar{Y}$is isometric to U , and hence $Is(X)$ is isomorphic to to a topological subgroup of $Is(U)$.

Every topological group G with a countable base is isomorphic to subgroup of $Is(X)$ for some separable Banach space X : There is a countable subset$A \in R \cup C^b(G)$ which generates the topology of G , and we can take for X the closed G-invarient linear subspace of $R \cup C^b(G)$ generated by A . We just saw that $Is(X)$ isomorphic to a subgroup of $Is(U)$. Thus, we have proved:

Theorem (3.6.2): Every topological group with a countable base is isomorphic to a topological subgroup of the group$Is(U)$.

Note that the group $Is(U)$ is Polish (= separable completely metrizable). Another example of a universal Polish group is the group$H(G)$ of all homeomorphisms of the Hilbert cube. To prove that every topological group$G$ with a countable base is isomorphic to a subgroup of $H(Q)$, it suffices to observe that:

1. G is isomorphic to a subgroup of $H(K)$ for some metrizable compact space K.
2. if K is compact and P(K) is the compact space of all probability measures on $K,$ there is a natural embedding of topological groups $H(K) \rightarrow H\big(P(K)\big)$;
3. If $K$is an infinite separable metrizable compact space, then $P(K)$ is homeomorphic to the Hilbert cube. The groups $Is(U)$and $H(Q)$ are not isomorphic and the group $H(Q)$ is not extremely amenable, since the natural of $H(Q)$ on Q has no fixed point.

The group $Is(U)$ is not Roelcke-precompact, to see that, fix $a \in U$ and consider the function $g \to d(a, g(a))$ from $Is(U)$ to R, where d is the metric on U. This function $L \wedge R$ uniformly continuous and unbounded, hence the Roelcke uniformity$L \wedge R$ is not precompact. We slightly modify the space U , in order to obtain a Roelcke –precompact group of isometries.

Let U_1be the "Urysohn universal metric space in the of spaces of diameter ≤ 1" . This space is characterized by the following properties:U_1 is a complete separable w −homogenous metric space of diameter$\leq 1$is isometric to a sub-space ofU_1 . Let $G = Is(U_1)$. This is a universal Polish group. This group is Roelcke precompact. Let us describe the Roelcke compactifications $R(G)$ of G .

Consider the compact space $K \subset I^{U_1}$ of all non-expanding functions $f: U_1 \to I = [0,1]$. Then K is a G −space, so there is a natural map from $R(G)$ to the set $ExpK^2$ of all closed relations on$K$. It turns out this map is homeomorphic embedding.

There is a more geometric description of $R(G)$; it is the space of all metric spaces M of diameter 1, which are covered by two isometric copies of U_1 . More precisely, consider all triples $S = (M, i, j)$, where M is a metric spaces M of diameter 1 , $i : U_1 \to M$ and $j : U_1 \to M$ are isometric embeddings, and $M = i(U_1) \cup j(U_1)$. Every such triples gives rise to the function $P_s : U_1 \times U_1 \to I$ defined by $P_s(x, y = d(i(x), j(y))$ where d is the metric on M. The set θ of all functions P , that arise in this way, is a compact subspace of $I^{{U_1}^2}$, and $R(G)$ can be identified with θ. Elements of G correspond to triples (M, i, j) such that $M = i(U_1) = j(U_1)$.

The space $R(G)$ has a natural structure of an ordered semi group. If $R(G)$ is identified with a subset of $ExpK^2$, then $R(G)$ happens to be closed under composition of relations, whence the semi group structure, and the order is just the inclusion. If $R(G)$ is identified with θ , then the order is again natural, and the semi group operation is defined as follows: If $p, q \in \theta$, the product of p and q in θ is the function $r : {U_1}^2 \to I$ defined by

$r(x, y) = inf(\{p(x, z) + q(z, y)\}) \cup \{I\} , x, y \in U_1$.

There is a one-to one correspondence between idempotents in $R(G)$ and closed subsets of U_1 .

Theorem (3.6.3): The universal Polish group $Is(U)$ is minimal.

Thus, every topological group with countable base is isomorphic to a subgroup of a minimal Roelcke-precompact Polish group. More generally, every topological group is isomorphic to a subgroup of a minimal group of the same weight.

Every topological group G is isomorphic to a subgroup of $Is(X)$ **,** where$X$ is a complete w-homogenous metric space of diameter 1, which is injective with respect to finite metric spaces of diameter 1, and for every such X the group $Is(U)$ is Roelcke-precompact and minimal. The uniqueness of X is lost in the one separable case, and it is not known whether there exists a universal topological group of a given uncountable weight.

4 Conclusion

We saw that the topological groups divide to many parts, like abelian and non-abelian topological groups and topological transformation groups, and plays a very important part in pure mathematics. There are many parts of topological groups' compactifications, and they may be useful for studying the group G itself.

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Assessment of Drinking Water Quality at Source and Point of use of Alazhari City

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Abstract

The present paper is aimed to assess the drinking water quality at source (ground water wells) and the point of use (Taps) at Al-Azhari city. The physicochemical and biological analysis of drinking water was carried out and results were compared with permissible limits established by the World Health Organization (WHO) and the Sudanese Standards and Metrology Organization (SSMO). The parameters subjected to the study were pH, TDS, EC, hardness, alkalinity, chloride, Mg, Ca, K, Na as well as bacteriological analysis. All the physicochemical parameters of drinking water samples collected from Al-azhari city are within permissible limit set by WHO and SSTO. Tap6 recorded relatively higher conductivity value (1040 µS/cm). The pH of the water ranged from 6.6 to 8.3 and TDS varied between 145.6 to 728 mg/l, EC varied between 224 and 1040 µS/cm, hardness varied between 84 and 262 mg/l, alkalinity varied between 20 to 576.2 mg/l, chloride varied between 9 and 99.93 mg/l. The heavy metals concentrations were 0.004 to 45.67, 18.4 to 68.93, 0.1 to 12.81, and 1.1 to 190 mg/l for Mg, Ca, K and Na respectively. The microbiological analysis had shown that all household waters except Tap2 were contaminated with E-coli. The absence of E. coli in wells 4, 5 and 6 and its presence in Taps4, 5 and 6 concluded that water in its way to households carry pollution through water distribution system.

Keywords: drinking water, water quality, standards, bacteriological and physiochemical parameters, Alazhari city.

تقييم جودة مياه الشرب من المصدرومن نقاط الاستخدام بمدينة الأزهري

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مُسْتَخْلَص

تهدف هذه الورقة لتقييم جودة مياه الشرب من المصدر (مياه الآبار) وعند نقاط الاستخدام (الحنفيات) لمدينة الأزهري. تم تحليل الخواص الفيزيوكيميائية والبيولوجية ومقارنة النتائج مع القيم المسموح بها في معايير منظمة الصحة العالمية والمعايير السودانية. المعاملات التي خضعت للدراسة هي التركيز الهيدروجيني والمواد الذائبة الكلية، الموصلية الكهربائية، العسر، القاعدية، الكلورايد، الماغنيسيوم، الكالسيوم، البوتاسيوم، الصوديوم بالإضافة إلى التحليل البكتريولوجي. كل الخواص الفيزيوكيميائية لعينات مياه الشرب المأخوذة من مدينة الأزهري في الحدود المسموح بها وفقاً لمعايير منظمة الصحة العالمية والمعايير السودانية. الموصلية الكهربائية للحنفية 6 أعلى قليلاً من الحدود المسموح به (1040 μS/cm). حدود التركيز الهيدروجيني 6.6-8.3 ، المواد الذائبة الكلية 145.6-728 ملجم / لتر، EC 224-1040 ميكرو سيمنز/ سم ، العسر يتراوح بين 84 و 262 ملجم/ لتر ، القاعدية 20 إلى 576.2 ملجم / لتر ، الكلورايد يتراوح بين 9 إلى 99.93 ملجم / لتر ، تركيز المعادن الثقيلة 0.004 إلى 45.67، 18.4 إلى 68.93 ، 0.1 إلى 12.81 ، 1.1 إلى 190 ملجم /لتر للماغنيسيوم ، الكالسيوم ، البوتاسيوم ، الصوديوم على الترتيب. التحليل المايكروبيولوجي أوضح أن كل المياه المأخوذة من المنازل ملوثة بالأيكولاي ماعدا الحنفية 2. غياب الأيكولاي في الآبار 4 ، 5 ، 6 و وجوده في الحنفيات 4 ، 5 ، 6 يوضح أن سبب التلوث هو الشبكة الناقلة للمياه.

كلمات مفتاحية:مياه الشرب، جودة المياه، المعايير، المعاملات البكتريولوجية والفيزيوكيميائية ، مدينة الأزهري.

Introduction

Water is the single most important substance known in the world it is elixir of life without its life is not possible, it represents a fundamental requirement for all life activities, it is essential to man, animals and plants.That water intended for human consumption must be free from chemical substances and micro-organisms in amounts which would provide a hazard to health is universally accepted (Eshraga, 2005). Water quality is the physical, chemical, and biological characteristics of water in association to the set of standards. These parameters directly related to the safety of the drinking water to human use.

One of the major types of monitoring is a research monitoring, which may be defined as: Measurements specifically related to research investigations (McNelis, 1973, and FAO, 1979). McNelis (1973) reported that monitoring is a necessary element of water quality considerations providing quantitative and qualitative data on existing circumstances and trends. As a result, water resources management (WRM) has been undergoing a change worldwide, moving from a mainly supply-oriented, engineering-biased approach toward a demand-oriented, multisectoral approach, often labeled integrated water resources management. (Daniel P. Loucks, 2016). The main objective of this study is to assess the physicochemical and the microbiological characteristics of drinking water at source and point of use from selected points of Alazhari city and to compare the results obtained with the WHO and SSMO standards.

Study Area

Alazhari city is located at the southern part of greater Khartoum city, Khartoum city the capital of Sudan is located in the central part of the country the state lies between longitudes 31.5 to 34 °E and latitudes 15 to 16 °N. Khartoum state is surrounded by river Nile State in the north-east, in the north-west by the Northern State, in the east and southeast by the states of Kassala, Qadarif, Gezira and White Nile State, and in the west by North Kurdufan. The weather is rainy in the fall, and cold and dry in the winter. Average rainfall reaches 100–200 mm in the north-eastern areas and 200–300 mm in the north-western areas. The temperature in summer ranges from 25 to 40 °C from April to June, and from 20 to 35 °C in the months of July to October. In winter, the temperature declines gradually from 25 to 15 °C between March and November. Wells under investigation were drilled in Al-azhari and Al-salma suburban communities (Figure 1).

Figure1: Location of the sampling stations of the wells

Methodology

Samples were collected from 12 locations; 6 from the source (groundwater wells) and other 6 from point of use (Taps). All samples were analyzed for 10 parameters, namely pH, electrical conductivity, TDS, total hardness, total alkalinity, chloride, calcium, magnesium, potassium and sodium. For the determination of coliform bacteria in water samples, the multiple tube technique or Most Probable Number (MPN) technique was carried out. In this study, all laboratory tests were carried out in accordance with Standard Methods[1]. Analysis of samples has been done at laboratories of the department of water and environmental engineering, Sudan University of science and technology, and UNESCO chair, Omdurman Islamic University.

Results and Discussion

The detailed respective analysis of physiochemical and biological quality parameters to ground water samples and the point of use are discussed as following:

pH

The pH value of water is a measure of acidity or alkalinity. The range goes from 0-14, with pH is 7 being the solution is neutral. When the concentration of hydrogen ions exceeds that of hydroxide ions, a pH value is less than 7 and hence water is acidic. Conversely, when the concentration of hydroxide ions exceeds that of hydrogen ions, the water is alkaline and has a pH value greater than 7. In this study, the results of the pH analysis of groundwater and tap samples are presented in Table 1. From the Table 1 the data showed that the pH of the water ranged from 6.6 at GW5 to 7.6 at GW4 and GW6 while levels recorded in tap waters ranged from 7.1 at Tap1 and Tap5 to 8.3 at Tap6.

Table1: pH at different sampling wells and taps:

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
pH	7.4	7.3	7.4	7.6	6.6	7.6	6.5-8.5	6.5-8.5
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	7.1	7.2	7.6	7.4	7.1	8.3		

All pH values are within the WHO and SSMO acceptable limit. The acceptable range of pH for drinking water quality is 6.5 to 8.5. The pH of drinking water has no immediate direct effects on human health but has some indirect health effects by bringing changes in other water quality parameters such as solubility of metals and survival of pathogen (Zabed *et al*., 2014).

Total Dissolved Solids(TDS)

Total Dissolved Solids (TDS) are inorganic compounds and small amounts of organic compounds that are dissolved in water. These solids include minerals like calcium, magnesium, sodium, salts and some traces of organic compounds such as decaying plant and animal matter. Particle is considered dissolved if it can pass through a filter of 2.0 micron size (1 micron = 1/1000 of a millimeter). Particles that are larger than 2 microns are called a suspended solid.

Table 2: TDS at different sampling wells and taps

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
TDS (mg/l)	377	306.8	275.6	345	275.6	291	1000	1000
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	386	276.3	145.6	275.6	221.6	728		

From the Table 2 the TDS values were found to be varied from 275.6 at GW3 and GW5 and 377 mg/l at GW1. Regarding tap water, the obtained results varied between 145.6 to 728 mg/l at Tap3 and Tap6 respectively. All water samples drawn from wells and taps were found within safe limits of WHO and SSMO guidelines. According to World Health Organization, TDS concentration of 1000 mg/litre is considered acceptable for water consumers; high concentration of TDS often has a bad taste and high-water hardness. A very low concentration of TDS may also be unacceptable because of its flat, insipid taste of water, the United States guidelines for TDS is 500 ppm (mg/l). EPA in USA includes TDS as secondary standards (SMCL) (Secondary Maximum Contaminant Levels).

Electrical conductivity EC

Conductivity is a measure of the ability of water to conduct electrical current; it is directly related to the total dissolved salt content of water. In this study EC ranged between 424 to 628 μS/cm at

GW5 and GW4 respectively for samples collected from wells, and ranged from 224 to 1040 µS/cm at Tap3 and Tap6 respectively for samples collected from tap water (Table 3). All water samples were found within the safe limits of SSMO guidelines (1000µS/cm) but samples collected from GW4, Tap1 and Tap6 were exceed the limit of WHO guidelines (500 µS/cm).Tap6 recorded relatively higher conductivity value (1040µS/cm), this value is greater than the WHO and SSMO guidelines (500 and1000µS/cm respectively), this might be due to the high level of soluble salts such as carbonates, chlorides, sulphates and nitrates and cations such as potassium, magnesium, calcium and sodium. It is well known that the conductance of water increases with salts and total dissolved solids (Meybeck, 1997)

Table 3: Electrical conductivity at different sampling wells and taps

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
EC (µS/cm)	460	472	424	628	424	451	500	1000
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	776	425	224	424	341	1040		

Hardness

Hardness is defined as the total concentration of calcium and magnesium cations contained in water. The hardness of water is mainly due to the presence of inorganic compounds such as carbonates, bi-carbonates, chlorides and sulphates of calcium and/or magnesium in dissolved form picked up from rocks and soils. In this study the hardness values of the source ranged between 84 to 262 at GW3 and GW6 respectively (Table 4). Regarding samples collected from taps the harness concentration varied between 135 to 220 mg/l at Tap4 and Tap1 respectively (Table 4). Several epidemiological investigations over the last 50 years have demonstrated a relation between risk for cardiovascular diseases and drinking water hardness or its content of magnesium and calcium (WHO, 1984). The total hardness level is not supposed to exceed 500 mg/l according to WHO and SSMO standards, hence all water samples drawn from source and taps meet WHO and SSMO maximum allowable levels. Based on Thomas classification (1953), it can be said that, all samples may be classified as hard water (180 mg/l and above), except GW2, GW3, Tap2, Tap4 and Tap6 may be classified as medium hard (60 to 179 mg/l).

Table 4: hardness at different sampling wells and taps

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
Hardness (mg/l)	215	172	84	216	236	262	500	500
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	220	164	188	135	184	136		

Alkalinity

Alkalinity is defined as the capacity of water to neutralize acid. Alkalinity of water is mainly caused by the presence of hydroxide ions (OH^{-}), bicarbonate ions (HCO^{3-}), and carbonate ions (CO_3^{2-}), or a mixture of these ions in water. Bicarbonates represent the major form of alkalinity.In higher alkalinity water, more acid can be added without considerable change in the pH and water of low alkalinity needs less acid to change pH.

Table 5: Alkalinity at different sampling wells and taps

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
Alkalinity (mg/l)	190	358.87	336.1	196	576.2	176	1000	1000
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	185	54	278.9	20	164.9	439.2		

The EPA Secondary Drinking Water Regulations (called a secondary maximum contaminant level or SMCL) limit alkalinity only in terms of total dissolved solids. In this study the alkalinity values varied between 176 and 576.2 mg/l in wells GW6 and GW5 respectively and varied between 20 and 439.2 mg/l (Table 5), these findings were within the range suggested by WHO and SSMO standards. Strong alkaline water has an objectionable "soda" taste. The high levels of either acidity or alkalinity in water may be an indication of industrial or chemical pollution.

Chloride

Table 6: Chloride at different sampling wells and taps:

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
Chloride (mg/l)	85.97	99.93	39.98	36	69.98	9	250	250
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	97.9	63.98	29.99	81.9	55.98	63		

Chloride occurs naturally in groundwater, streams due to rock containing chloride, other sources might be agricultural runoff, domestic and industrial wastewater. Chloride ions Cl^{-} in drinking water do not cause any harmful effects on public health. Small amounts of chlorides are essential for ordinary cell functions in animal and plant life but high concentrations may indicate contamination by sewage or animal waste, high chloride concentration in freshwater (more than

250 mg/L) can also cause an unpleasant salty taste for most people. WHO and SSMO standards for public drinking water require chloride levels that do not exceed 250 mg/L. In this study, the levels of chloride for the wells varied from 9 to 99.9 mg/l at GW6 and GW2, respectively. The levels for tap water varied from 29.99 to 97.9 mg/l at Tap3 and Tap1 respectively as shown in Table 6. Therefore, all wells and tap water samples were observed to meet WHO and SSMO acceptable levels.

Heavy metals

Table 7: Heavy metals at different sampling wells and taps:

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
Mg (mg/l)	3.42	28.25	14.18	28.8	45.67	16.32	150	150
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	0.004	28.17	25.65	25.12	34.94	9.72		
Ca (mg/l)	60.92	41.62	26.65	38.4	48.1	18.4	200	200
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	68.93	48.1	25.65	32.06	40.24	38.4		
K (mg/l)	7	2.05	12.81	4.1	0.25	3	NS*	12
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	7	0.28	10.67	0.32	0.1	9.728		
Na (mg/l)	75.45	2.84	2.7	38	3.27	41.5	200	200
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	75.45	1.98	3.24	1.1	63.17	190		

* No Standard

The term heavy metals refer generally to any metallic chemical element that has a relatively high density; heavy metals are natural components of the Earth's crust. Heavy metals can enter a water supply by industrial and domestic waste, or even from acidic rain breaking down soils and releasing heavy metals into surface water and groundwater. Some heavy metals are essential to human health if their level remains within the specified range recommended by WHO. They maintain the metabolism of the human body. However, at higher concentrations they can pose a threat to human health and poisoning. In this study, four heavy metals, magnesium (Mg), calcium (Ca), potassium (K), and sodium (Na) were determined by atomic absorption photometers in wells and tap waters of Al-azhari city and results are shown in Table 7. The data showed that the magnesium concentration of groundwater ranged from 3.42 to 45.67 mg/l at GW1 and GW5 respectively and for tap water the concentration varied from 0.004 to 34.94 mg/l at Tap1 and Tap5 respectively. The results indicate the water is free of magnesium pollution according to WHO and SSMO maximum allowable levels (150 mg/l). Calciumlevels found to be within the safe limits of WHO and SSMO standards (200mg/l). It varies between 18.4 and 60.92 mg/L at GW6 and GW1 respectively and for tap water, the concentration varied from 25.65 to 68.93 mg/L at Tap3 and Tap1respectively. Potassium concentrations in water samples from wells ranged from 0.25 to 12.81 mg/L at GW5 and GW3, respectively, whilst levels recorded in tap waters ranged from 0.1 to 10.67 mg/L at Tap5 and Tap3 respectively. Concentrations of potassium were found to be within SSMO maximum allowable levels (12 mg/l) except slight excessive value at GW3 (12.81 mg/l). No standard or guideline has been established by WHO for potassium parameter. The **so**dium concentrations recorded in samples from wells ranged from 2.7 to 75.45 mg/L at GW3 and GW1 respectively. Samples from point of use recorded sodium values that ranged from 1.1 to 190 mg/L at Tap4 and Tap6, respectively. All sodium ion concentration had levels within the acceptable WHO guidelines value of 200 mg/l.

Microbiological quality

Microbiological characteristics are used to describe pathogenic microorganism such as bacteria, protozoa and viruses. Pathogen is any living organism that causes disease. Bacterial examination of water is very important, since it indicates the degree of pollution. Microbiological analysis of the water samples was performed by determination of most probable number technique of coliform bacteria in 100 ml of water sample. Because animals can transmit pathogens that are infective to human, the presence of E. coli must not be ignored (WHO, 1993). E. coli shouldn't be detected according to the WHO and SSMO standards. The results of the microbiological analysis of groundwater and tap samples are presented in Table 8. In this study, no coliform was detected from samples collected from well 2, 4, and 6. The study has confirmed the presence of E-coli in well1, 3, and 5. The highest count was found in well 3 (1100/100 ml) showing that the great risk of using water of this well, it is not suitable for human consumption; action should be taken by authority. The second highest number of bacteria was found at Tap5 (460/100ml). Although well 1, Tap1, Tap4, and Tap6 showed a smaller number of bacteria (3-43 /100 ml) but same attention should be taken to prevent bacteriological pollution. The microbiological examination of water in this study showed that the water network distribution system is the source of contamination of Tap4, 5 and 6, therefore, detailed studies should be undertaken along the distribution lines starting from their source to the households to find out the actual points of contamination and their sources in distribution system.

Table 8: Microbiological analysis of the water samples of wells and Taps:

Parameter	GW1	GW2	GW3	GW4	GW5	GW6	WHO	SSMO
E. coli MPN/100	3	0	1100	0	0	0	0	0
	Tap1	**Tap2**	**Tap3**	**Tap4**	**Tap5**	**Tap6**		
	3	0	1100	4	460	43		

Conclusion

All the physicochemical parameters of drinking water samples collected from Al-azhari city are within permissible limit set by WHO and SSTO, slight excess of EC recorded at Tap6. Regarding the biological parameters, a high microbial indicator count was detected in the water source well 3, water from this well is not suitable for human consumption. All household water samples except Tap2 were found to be unsatisfactory as they contained coliforms. The absence of E. coli in wells 4, 5 and 6 and its presence in Taps 4, 5 and 6 concluded that water in its way to households carry pollution through pipes and other distribution installations.

Recommendations

1. Water from the source GW3 is highly contaminated with E. Coli, this well is not suitable for human consumption, an urgent action should be taken by authority for proper drinking water treatment.
2. Detailed studies should be undertaken along the distribution lines starting from the source to the households to find out the actual points of contamination and their sources in the distribution networks and more study of parameters that have an effect in water quality.
3. Before a new source of drinking water supply is selected, it is important to ensure that the quality of the water satisfactory for drinking.
4. The old pipelines of the network must be replaced by a new one and distribution system must be periodically treated by chloride to prevent contamination by microorganisms.
5. Sources of ground water wells should be sited a constructed so as to be protected and to prevent public access and animals.
6. The statistical tools developed, for further studies required to be extended to larger areas covering several water resources and proposing water quality models. In addition to

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Prevalence of Eimeria Species in Goats in EdDamer Locality, River Nile State, Sudan

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Abstract

Monthly collection of fecal samples was made from 682 healthy goats in EdDamer Locality, River Nile State, over twelve months period. The species of Eimeria involved in subclinical cases were identified. 82% out of examined goats were found to be positive to Eimeria species and seven Eimeria species were recorded, these were, E. arloingi (96.24%), E. alijevi (64.79%), E. christenseni (54.46%), E. hirci (27.2%) E. ninakohlyakimove (24.88%), E. jolchijevi (16.43%), and E. apsheronae (14.84%). The prevalence of Eimeria species differed significantly (P<0.05). The prevalence of all identified species was higher in young age groups, but no significant difference was observed (p>0.05), between the three examined age groups. The prevalence of species during different seasons did not also seem to vary significantly (P>0.05), but the prevalence of all identified species was higher during Summer.

Keywords: Goats, Coccidia, Eimeria, EdDamer Locality.

انتشار طفيل الايمريا في الماعز بمحلية الدامر، ولاية نهر النيل، السودان

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مُسْتَخْلَص

اجريت هذه الدراسة في محافظة الدامر بنهر النيل للتعرف على انواع طفيل الايميريا التي تصيبالماعز. تم جمع 682 عينة براز من ماعز سليمة ظاهريا. كانت نسبة الاصابة 82% وقد تم التعرف على سبعة انواع من الايميريا هي ايميريا ارلويني (96%)، ايميريا اليجافي (64%)، ايميريا كرستنسني (54%)، ايميريا هيرسي (27%)، ايميريا نيناكوهليا -كيموفي (24%)، ايميريا جولشيجافي (16%) وايميريا ابشيروني (14%). نسبة انتشار كل الانواع كانت عالية في الماعز الصغيرة مقارنة بالكبيرة ولكن ليس هناك اختلاف ذو دلالة في الاصابة بين الصغار والكبار. لاتوجد دلالة إحصائية لانتشار الانواع خلال فصول السنة.

كلمات مفتاحية: الماعز، الكوكوسيديا، الإيميريا، محلية الدامر.

Introduction

Coccidia belonging to the genus *Eimeria* show a wide distribution and occur in all kinds of environment, causes a disease called coccidiosis (Levine and Iven, 1986). Several species of *Eimeria* occur commonly in livestock worldwide, and fifteenspecies were described earlier in domestic goats, these were: *E. arloingi, E alijevi, E. Christenseni, E. apsheronae, E. caprina, E. caprovina, E. hirci, E. jolchijevi, E. kochari, E. ninakohlyakimovae, E. pallid, E. punctata, E. skerjabini, E. minasensis and E. africiensis* (Levine and Iven, 1986). Moreover, More *et al.,* (2015) described recently three new species in goats in India, these were*: E. straightaatus, E susheelensis* and *E leafii.* Many surveys and studies on *Eimeria* species of goats in the Sudan were conducted (Osman, 1988; Elrabie and ElHussein, 2000; Fayza *et al*., 2004; Yousif, 2010). Earlier, experimental infection of Sudanese goats with either *E. arloingi* alone (El Gazuli *et al.*, 1979) or a mixture of *E. arloingi* and *E. parva* (Shommein and Osman, 1980) showed the ability of these species of *Eimeria* to infect indigenous goats and cause clinical coccidiois disease in Sudan. Halima, (1988) found *Eimeria* oocyst in 81% of fecal samples obtained from apparently healthy goats in Khartoum State, Sudan and two species viz *E. arloingi* (73%) and *E. parva* (27%) were identified in these animals. Fayza *et al*. (2004) stated that an acute coccidiosis was reported in three to six months old male kids in Khartoum State, all were naturally infected with five *Eimeria* species which were identified as *E. christenseni* (50%) *E. arloingi* (32%) *E. hirci* (8%), *E. ninakohlyakimovae* (6%), and *E. alijevi* (4 %). In EdDamer Locality, Northern Sudan, coccidiosis ranked highest among diseases that affect goats (El Ghali and El Hussein, 1995) and five species of *Eimeria* viz, *E. arloingi* (90%), *E. alijevi* (49%), *E. hirci* (33%), *E. christenseni* (24%), and *E. ninakohlyakimovae* (10%), were described in 21 goats with clinical coccidiosis (El Rabie and El Hussein, 2000) in that area. However, no study was made regarding *Eimeria* infections in apparently healthy goats. The River Nile State lies in the northern part of the Sudan between longitude 32 C^o–35 C^o east and latitude 16 C^o–22 C^o north. It is characterized by its desert or almost semi–desert climate. In this state the weather is cold in winter and hot or very hot in summer. This state enjoys a considerable area for agriculture and natural pasture which is almost confined to the banks of the River Nile and Atbara River. The state was divided into seven Localities. EdDamer Locality represents the capital of the state (Anon, 1996). The present study was aimed to determine the species of *Eimeria* involved in subclinical cases (apparently healthy goats), and to study the effect of some factors (age, seasons) on the prevalence and intensity of these species in EdDamer Locality, River Nile State, northern Sudan.

Material and Methods

Collection of samples

A total of 682 fecal samples were collected monthly from various age groups of apparently healthy goats during twelve months (one year) in EdDamer Locality. The fecal samples were directly collected from animal's recta (5-10 grams from each animal) in small plastic bags and kept in refrigerator at 4 C^o at Atbara Regional Veterinary Research Laboratory until tested.

Parasitological examination

The presence of *Eimeria* oocysts in fecal samples was examined by the cover–slip flotation method using saturated NaCl solution as the flotation medium (Anon, 1986). Briefly 3 grams of each fecal sample were weighted out using a sensitive balance and put in a 50 ml beaker. 42 ml of water was then added, mixed thoroughly and poured into a 100ml glass beaker through a strainer. The 50ml glass beaker was rinsed with 8 ml of water and the total fluid poured into four 15ml conical tip centrifuge tubes. After centrifugation at 1500 rpm for 5 minutes the supernatant was decanted and a saturated NaCl solution added to the sediment, until the tube was about half full. The content of each test tube was thoroughly mixed with a wooden applicator stick, and more NaCl solution was added until a convex level was formed at top of the tube. A glass coverslip was placed on top of each tube and left for 30 minutes. Then each glass cover slip was briskly lifted up and placed on a clean glass slide, the entire area under each coverslip was examined under a binocular microscope at ×10, and×40.

Sporulation of *Eimeria* oocysts

About 2 grams of the positive samples with *Eimeria* were mixed in tap water and filtered through 300 µm mesh screen. The filtrate was transferred into a cylinder and allowed to stand overnight. The supernatant was discarded and the sediment was centrifuged at 1500 r.p.m for 5 minutes and finally the sediment was suspended in a shallow layer of 2.5 % (w/v) potassium dichromate solution ($K_2Cr_2O_7$) in Petri dishes and was left to sporulate a laboratory temperature (25-27C^o) under aeration until sporulation was completed (Osman *et al.*,1990; El Rabie and El Hussein, 2000).

Identification of *Eimeria* species

Identification of the sporulated oocysts was made to determine *Eimeria* species. The species were identified based on the morphological characteristics of their oocysts and sporocysts (size, shape, colour, presence or absence of micropyle and its cap, presence or absence of residual body, polar granules and stieda bodies), (Levine and Iven, 1986; Heidari *et al.*, 2014). Also, the measurement of oocysts and sporocysts was done, for 20 - 50 sporulated oocysts which were randomly selected from each samples using an eyepiece micrometer and compound binocular microscope (Olympus, Japan), and identified to determine the percentages of each *Eimeria* species (Wang *et al.*, 2010). 4800oocysts of *Eimeria* of goats (from 212goats) were described and identified to determine the prevalence of *Eimeria* species.

Statistical analysis

Statistical analysis was performed using the SPSS software package for social sciences version 16.0 for windows. The differences between the prevalence of speciesaccording to various factors were evaluated using chi-squire, t testand analysis of variance (ANOVA). Values of $P<0.05$, $P<0.01$ and $P<0.001$ were respectively considered as significant, moderately significant and highly significant.

Results

Seven *Eimeria* species were identified in fecal samples of goats and a significant difference ($P<0.01$) was observed between the prevalence rates of these species (mean 47.33% ± 3.04%). Their prevalence rates were as follows: *E. arloingi* (96.2%), *E. alijevi* (64.79%), *E. christenseni* (54.46%), *E. hirci* (27.2%), *E. ninakohlyakimovae* (24.88%), *E. jolchijevi* (16.43%) and *E. apsheronae* (14.84%), (Table1). The measurement of oocysts and sporocysts of isolated *Eimeria* species was varied from species to another (Table2).

Table 1: Prevalence of *Eimeria* species in goats in EdDamer Locality:

Species	Infected samples	%	P value
E. arloingi	538	96.20%	0.01
E. alijevi	362	64.79%	
E. christenseni	304	54.46%	
E. hirci	152	27.20%	
E.ninakohlyakimovae	139	24.88%	
E. jolchijevi	092	16.43%	
E. apsherona	083	14.84%	

Table 2: Measurements of Eimeria species recovered from goats in EdDamer Locality

	Size of oocysts in microns				Size of sporocysts in microns			
	Length		Width		Length		Width	
	Mean ±S	Range	Mean ±SD	Range	Mean ±S	Range	Mean ±SD	Range
E. arloingi	29.7 ±1.7	22-37	20.2 ±1.4	17.5-25	14.4 ±1.3	12.5-17.5	6.7 ±1.2	5.7-7
E christenseni	37.4 ±1.7	27-45	25.5 ±2.8	20-30	13.7 ±1.4	10-17.5	9.0 ±1.2	7-10.2
E. alijevi	23.7 ±1.7	15-25	15.2 ± 2.8	15-20	10.9 ±1.2	7.5-12.5	8.5 ±1.2	5.-10
E. hirci	23.7 ±2.9	8 -27	18.2 ± 1.6	15-22	9.1 ±1.8	7.5-12.5	6.2 ±1.2	5.1-7
E.ninakohlyak.	27.1 ±2.3	22-30	20.9 ± 1.6	17-22	12.9 ±1.9	10-15	6.4 ±1.4	5-10
E. jolchijevi	27.1 ±2.3	24-36	20.9 ± 0.9	10-15	12.1 ±1.7	11.2-15.6	6 ±0.7	5-9
E. apsheronae	28.8 ±2.8	25-32	20.5 ± 2.1	13-15	14.3 ±1.3	10 -15.6	8.9 ±1.2	7-10

Identification of a total of 4800 oocysts from 212 goats showed that the oocysts of *E arloingi*, represented the highest percentage encountered being 57%, while *E.hirci*, *E. ninakohlyakimovae*, *E. jolchijevi* and *E. apsheronae* were much less abundant (6%,3%, 2%, and 1% respectively), the percentage of oocysts of *E. alijavi*, *E. christenceni*, were 17%, 14%, respectively.

The present investigation also revealed that 5% of infected goats were infected with one *Eimeria* species (pure infection). These involved *E. arloingi* (3%), *E. alijave* (1.5%), *andE. chrisrenseni* (0.5%). The rest of the cases (95%) were infected with more than one *Eimeria* species, (multiple *Eimeria* species infection). Mixed infection by three *Eimeria* species being the most (40.85%)prevalent. Concurrent infection with *E. arloingi, E. alijavi* and, *E. chrisrenseni* (16%) and concurrent infection with *E. arloingi, and E. alijavi* (15%) represented the most prevalent types of multiple infections encountered in the tested animals. No multiple infection with all seven species of *Eimeria* were observed (Figure1). Table 3 shows the number of species in individual fecal specimen.

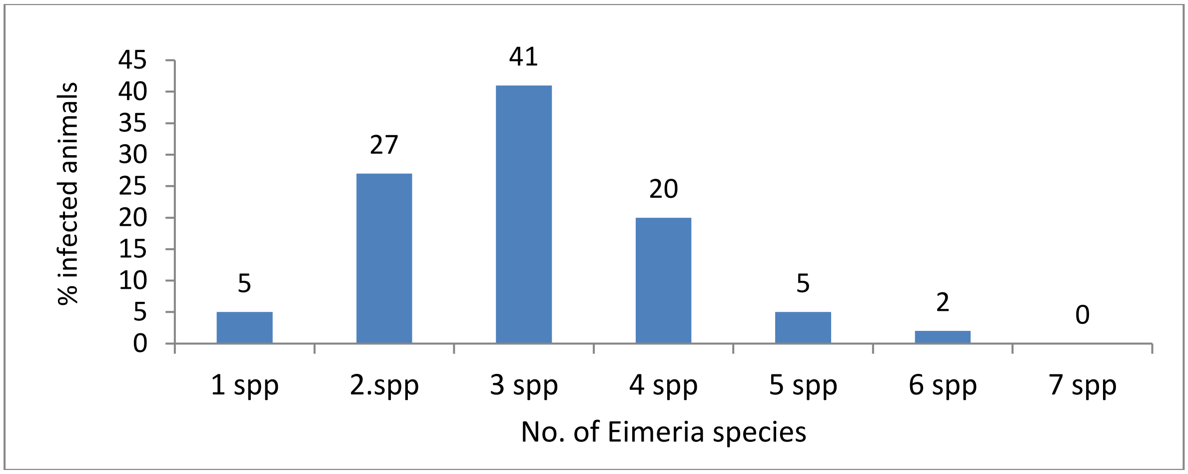

Figure 1: Multiple *Eimeria* species infection in goats in EdDamer Locality

Table 3: The percentage occurrence of goats *Eimeria* species and their number in individual fecal specimens

Eimeria species	**Occurrence**	**Number of *Eimeria* species in individual**						
		1	**2**	**3**	**4**	**5**	**6**	**7**
E. arloingi	96.24%	3.29%	27.33%	38.97%	19.25%	5.63%	1.88%	0
E. alijevi	64.79%	1.41%	15.2%	30.5%	12.21%	4.23%	1.88%	0
E.christense	54.46%	0.47%	8.9%	25.35%	14.8%	3.76%	1.88%	0
E. hirci	27.2%	-	2.35%	11.07%	4.45%	3.29%	1.88%	0
E. ninakohly	24.88%	-	-	7.04%	10.8%	5.16%	1.88%	0
E. jolchijevj	16.43%	-	-	5.16%	6.1%	3.76%	1.41%	0
E.apsherona	14.84%	-	0.94%	4.69%	6.1%	2.35%	0.76%	0
Total		**5.16%**	**27.23%**	**40.85%**	**19.25%**	**5.63%**	**1.88%**	**0**

Infections with *E. arloingi*, *E. alijavi*, and *E. chrestenseni* were high in all seasons, the infection rate ranged between 44% and 99%, while infections with *E apsheronae* and *E. jolchijavi* were lower, with infection rates ranging between 13% and 30%. High infection rates of all species were found during summer season (Figure 2), but no significant effect for seasons on the prevalence of *Eimeria* species was detected (P>0.05).

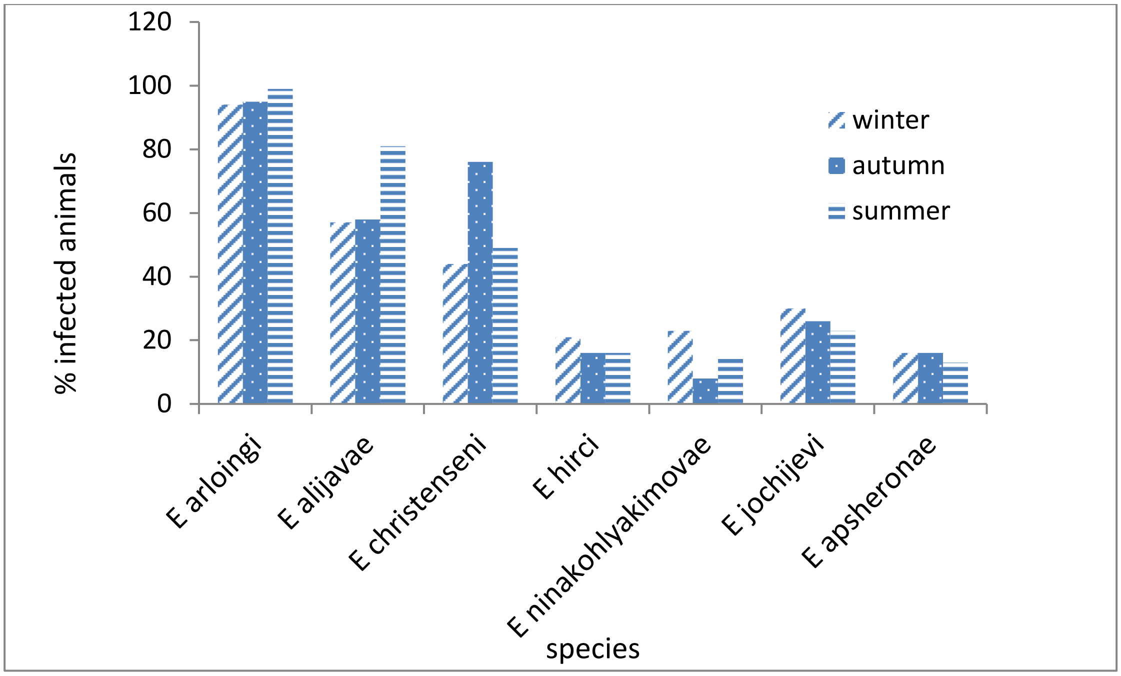

Figure2: Effect of seasons on the prevalence of *Eimeria* species in goats in EdDamer Locality

The infection with each *Eimeria* species differed according to the age of animal, but no significant effect was detected between the examined age groups (P>0.05). The infection with *E. arloingi, and E. alijevi* were high in all groups, the infection rate ranged between 59.1% and 99.4%, while the infection with *E. apsheronae* and *E. jolchijave* were low in all age groups, and ranged between 11 % and 27.5 %. The infection rates with *E. chrestenseni*, *E.hirci* and *E. ninakohlyakimovae* ranged between 19.5 % and 51% (Figure3).

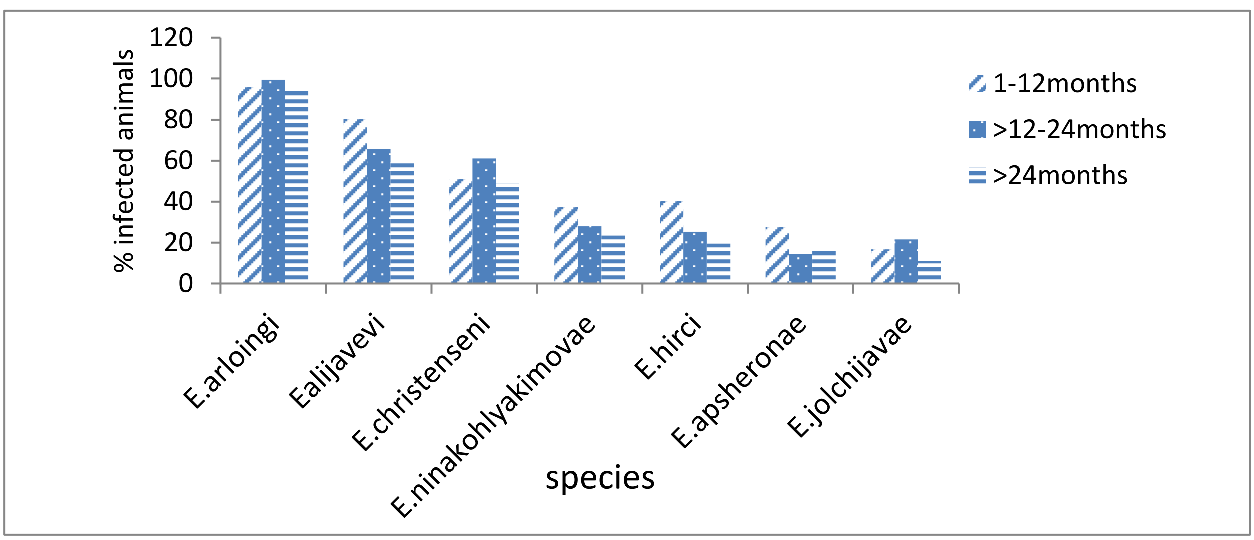

Figure 3: Prevalence of Eimeria species of goats in EdDamer Locality according to age

Discussion

In this investigation the *Eimeria* species infecting goats in EdDamer Locality were identified and effect of age and season on the prevalence of these species was studied. The overall infection rate with *Eimeria* was high (82%) in this province. Several works have been made to identify *Eimeria* species by several researchers worldwide (Kareem, 2015; More *et al*., 2015; ElShahawy, 2016; Mohamaden *et al.,* 2018). Also, many studies were made in many states of Sudan and various *Eimeria* species were recorded (Abakar, 1996; Shmaon, 2005; Abdel Wahab *et al*., 2008). Previous investigations in the world showed that one or more species of *Eimeria* are usually involved in infection of goats and sheep (Kareem, 2015; More *et al*., 2015). The present study revealed that seven *Eimeria* species are commonly found in fecal samples of goats (table1), this finding was similar to Mohamaden *et al.* (2018). However, ElRabie, and ElHussein (2000) identified five *Eimeria*species in clinically affected goats in EdDamer Locality and similarly Fayza *et al*. (2004) also identified five *Eimeria* species in diseased goats in Khartoum State. However, an earlier study by Halima, (1988) recorded only two species *(E. arloingi* and *E. parva*) of *Eimeria* in sampleswhich were obtained from 2000 goats in Soba area (Khartoum State). It should be noted, that while in the latter study fecal samples were obtained from goats confined to a limited area (Soba) our study covered a wider geographical area with varied ecological niches. Similar to our results ElShahawy, (2016) found seven species of *Eimeria* in goats in Upper Egypt, six of these species were identified in EdDamer Province, but *E. christenseni* was absent according to his study and *E. caprovina* was found instead of the former one. Moreover, similar findings were recorded in other hosts and more than two species were usually identified. Bakunzi *et al*. (2010) found seven species in South Africa in sheep. Mammadov (2014) found eight species in cattle, seven species in buffalo, and five species in chickens in Azerbigan, while, Yakhchali and Athari (2010) found that camels were infected with four *Eimeria* species in Tabriz region in Iran. These differences in prevalence of *Eimeria* species may be due to various sanitation efforts in the management programs by producers to control coccidiosis or due to differences in ecological and environmental conditions. In the current study *E arloingi*, and *E. alijavi* were the predominant species in goats (57% and 17%, respectively). while *E.* ninakohlyakimove, *E. jolchijavae* and *E. apsheronae* were much less abundant (3%, 2, and 1%, respectively) (fig 1). Common findings obtained from many investigations indicated that the percentages of oocysts of *E. arloingi, E. hirci, E. alijavi, and E. christenseni,* were high in goats in other parts of the world (More *et al*., 2015; Kareem, 2015). Explosive discharges of oocysts might occur giving rise to temporary dominance of some species over the coccidial population (Kareem, 2015).

Mixed infections in a single host were common findings in the world (More *et al*., 2015; Kareem, 2015), as noticed most of the infections reported here were of the mixed type and 95% of positive animals had from two to six *Eimeria* species in goat. This finding is consistent with the findings of other researchers in several parts of the world (Wang *et al.*, 2010; Bakunzi *et al.*, 2010; Khan *et al.*, 2011; Radifar, 2011; More *et al.*, 2011; More *et al.*, 2015; Kareem, 2015), similarly Abakar (1996) found that most naturally acquired coccidian infections in sheep in Sudan were mixed infections having between 4–6 species. Furthermore, Vercruysse (1982), stated that mixed infection in naturally acquired coccidiosis were the rule in 94% of goats examined in Senegal and involving 3–6 species of *Eimeria*. The present investigation revealed that 5% of goats were infected with one *Eimeria* species (pure infection) (fig 1). The rest of the cases (95%) were infected with more than one species of *Eimeria,* (mixed infection), concurrent infection with *E. arloingi, E. alijavi* and *E. christensni* (16%) represented the most prevalent types of multiple infection. Furthermore, no multiple infection with all seven species of *Eimeria* were observed. Moreover, oocysts of *E. arloingi* were the dominant (57%) oocysts encountered in all infections. Ayana *et al*., (2009) found that the maximum number of *Eimeria* species was four species per sample. Bakunzi *et al*. (2010) found that up to 5 species were recovered from individual specimens in goats while up to 4 species were recovered in sheep. Kareem, (2015) stated that 6.5% of infected sheep were infected with single species and 67.3% were mixed infection and the highest rate of mixed infection include six *Eimeria* species. In camels Yakhchali and Athari, (2010) observed that 6.12% of camels were infected with one *Eimeria* species.

Yakhchali and Zarei, (2008) reported that the infection rate of *Eimeria* species decreased due to the absence of rainfall, high temperature, and low relative humidity. Hence aggregation of animals of different age groups during cold season was considered as the most important factor that influence seasonal variation on the percentage of *Eimeria* species. In our study goats, infections with *E. arloingi*, *E. alijavi,* and *E. christenseni*, were highest in all seasons, the infection rate ranged between 44% and 99% while infection with *E apsheronae* and *E. jolchijavi* was low, the infection rate ranged between 13% and 30%. High infection rate was found during the rainy and cold seasons for the most species. These findings were similar to Yakhchali and Zarie (2008) who stated that although *Eimeria* species were presented in sheep through the year their percentage seemed to increase particularly during the late fall and whole winter.

In our study the percentage of infection with each species differs according to the age of animal and *Eimeria* species were more intense in young animals, this is similar to finding of Ibrahim and Afas (2013) who stated that the prevalence as well as intensity and diversity of *Eimeria* species were declined with host age, and *Eimeria* species were more intense in young animals than adults because the acquired immunity has been shown to causea decrease in infection of various *Eimeria* species with increase of host age.

Conclusion

This study was conducted on Caprine *Eimeria* species in EdDamer Locality, River Nile State, Sudan. The study results revealed that seven species were present in goats. Mixed infection with two species or more was the rule. No significant effect on the prevalence of *Eimeria* species was observed due to seasons or age.

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